

A NOTE ON THE MONGE-AMPÈRE TYPE EQUATIONS WITH GENERAL SOURCE TERMS

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ABSTRACT. In this paper we consider the generalised solutions to the Monge-Ampère type equations with general source terms. We firstly prove the so-called comparison principle and then give some important propositions for the border of generalised solutions. Furthermore, we design well-posed finite element methods for the generalised solutions with the classical and weak Dirichlet boundary conditions respectively.

1. INTRODUCTION

Let Ω be a bounded open convex domain in \mathbb{R}^d and $W^+(\Omega)$ denote the set of convex functions over Ω . Suppose that μ is a non-negative Borel measure in Ω and R is a locally integrable function in \mathbb{R}^d with $R(\mathbf{p}) > 0$ for any $\mathbf{p} \in \mathbb{R}^d$.

In this article, we mainly consider the generalised solutions to the Monge-Ampère type equations with Dirichlet boundary conditions. Firstly, for the Monge-Ampère type equations with the classical Dirichlet boundary condition, the generalised solution is defined as follows:

Definition 1.1. We call $u \in W^+(\Omega) \cap C(\bar{\Omega})$ a generalised solution to the classical Dirichlet problem of the Monge-Ampère equation if the following conditions hold:

$$\int_{\partial u(e)} R(\mathbf{p}) d\mathbf{p} = \mu(e) \quad \text{for any Borel set } e \subset \Omega, \quad (1.1a)$$

$$u = g \quad \text{on } \partial\Omega. \quad (1.1b)$$

Here $\partial u(e)$ denotes the sub-differential of u over e .

Furthermore, we also consider the Dirichlet problem with the weak boundary condition. In this case, we define generalised solution in the following way:

Definition 1.2. $u \in W^+(\Omega)$ is called a generalised solution to the Monge-Ampère equation with weak Dirichlet boundary condition, if it satisfies

$$\int_{\partial u(e)} R(\mathbf{p}) d\mathbf{p} = \mu(e) \quad \text{for any Borel set } e \subset \Omega, \quad (1.2a)$$

$$\limsup_{\Omega \ni \mathbf{x}' \rightarrow \mathbf{x}} u(\mathbf{x}') \leq g(\mathbf{x}) \quad \text{for any } \mathbf{x} \in \partial\Omega, \quad (1.2b)$$

and for any $v \in W^+(\Omega)$ satisfying (1.2), it holds:

$$u(\mathbf{x}) \geq v(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \Omega. \quad (1.3)$$

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The Dirichlet problems (1.1) and (1.2) mentioned above are natural generalisations of the following problem:

$$\begin{cases} \det(D^2u) = \mu & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \quad (1.4)$$

The boundary problem (1.4), arising from analysis and geometry, plays a very important role in the area of PDEs, and it has received considerable study since 1950's. This class of problems were firstly solved in the generalised sense by Alexandrov[1] and Bakelman[2], where they defined the generalised solution in the same way as Definition 1.1 and they proved the existence and uniqueness of generalised solutions (see also [11]).

With additional assumptions that $d\mu = f dx$ and $0 < f \in C(\overline{\Omega})$, Caffarelli [5] showed the equivalence between the notions of generalised solutions and viscosity solutions (also see [11]). Furthermore, for this special case, there are various regularity results on generalised solutions if f , $\partial\Omega$ and g are certain regular. For the global regularity, the results were established by Cheng and Yau[7, 8], Ivochkina[12], Krylov[13, 14, 15], Caffarelli, Nirenberg and Spruck[6], Wang[19], Trudinger-Wang[18] and Savin[17]. As for the interior regularity of generalised solutions, we refer readers to the work of Caffarelli[4, 5], De Phillipis and Figalli[9] and De Phillipis, Figalli and Savin[10].

For general R and μ , due to the lack of the regularity assumptions, it is impossible to relate generalised solutions to viscosity solutions and thus in the general case, the study on generalised solutions is totally different from that on viscosity solutions. Especially, the regularity problem on generalised solutions becomes tricky. For boundary problems (1.1) and (1.2), the solvability was firstly studied by Bakelman (see [3]). In his work, the results on existence and uniqueness of generalised solutions to (1.1) and (1.2) were established with more suitable assumptions on R and μ .

In our work, the main goal is to study the properties of generalised solutions to the Dirichlet problems (1.1) and (1.2). More precisely, we organise the remaining content of this article as follows:

Section 2 is devoted to proving Theorem 2.1, the so-called comparison principle, an important tool for studying the generalised Monge-Ampere type equations. Theorem 2.1 is a generalisation of [3, Theorem 10.1] and our proof is also inspired by the work of Bakelman. However, at some crucial steps, there are some gaps in the proof of [3, Theorem 10.1] (we shall explain these in details in section 2). To overcome this difficulty, two important lemmas: Lemma 2.2 and Lemma 2.3 would be given to derive Theorem 2.1.

In section 3, we consider boundary behaviour of convex function over Ω . We give Lemma 3.2, which describes some important proposition on the border of convex functions.

In section 4, we show convergence properties of a sequence of convex functions and this section consists of two parts: convergence of a sequence of convex functions inside convex domain and convergence of a sequence of borders of convex functions.

In section 5 and 6, a finite element method to the problem (1.1) would be given and shown to be well-posed. Furthermore, we prove that this finite element method converges to the exact solution to (1.1).

In section 7, a well-posed finite element method would be designed for (1.2) and we prove the convergence of such method to the exact solution to (1.2).

2. COMPARISON PRINCIPLE

In this section, we prove the comparison principle:

Theorem 2.1. *Assume that z_1 and $z_2 \in W^+(\Omega)$ satisfy the conditions: $z_1 \in C^0(\overline{\Omega})$ and $z_1(\mathbf{x}) \geq \limsup_{\mathbf{x}' \rightarrow \mathbf{x}, \mathbf{x}' \in \Omega} z_2(\mathbf{x})$ for any $\mathbf{x} \in \partial\Omega$. If the following inequality holds:*

$$\int_{\partial z_1(e)} R(\mathbf{p}) d\mathbf{p} \leq \int_{\partial z_2(e)} R(\mathbf{p}) d\mathbf{p} \quad \text{for every Borel set } e \subset \Omega,$$

then $z_1(\mathbf{x}) \geq z_2(\mathbf{x}), \forall \mathbf{x} \in \Omega$.

Remark: Theorem 2.1 is important not only for showing uniqueness of the solution but also for proving existence of the generalized solution with weak Dirichlet boundary condition in Definition 1.2 (see (7.5)). There are several important facts we need to point out:

(1) For the classical Monge-Ampère equations (where $R \equiv 1$), the comparison principle can be proved by the Brunn-Minkowski inequality (see [4, Theorem 1.4.6]). However, for general $0 < R \in L^1_{\text{loc}}(\mathbb{R}^d)$, that inequality is not available. Thus the argument of the proof of comparison principle for classical case cannot be applied to the general case.

(2) Theorem 2.1 is basically the same as [3, Theorem 10.1], and our proof is inspired by that of [3, Theorem 10.1]. However, the proof of [3, Lemma 10.2] lacks of some essential details. In the following we shall explain those gaps:

Firstly, one essential argument in the proof of [3, Lemma 10.2] states as follows: assume that $z_1, z_2 \in W^+(\Omega)$, Q is an open set satisfying $\overline{Q} \subset \Omega$. If there is $\mathbf{p} \in \mathbb{R}^d$ such that \mathbf{p} is contained in the boundary of $\partial z_2(Q)$, $\mathbf{p} \in \text{Int}(\partial z_1(Q))$, and $\partial z_2(Q) \subset \partial z_1(Q)$, then

$$\int_{\partial z_1(Q)} R(\mathbf{p}) d\mathbf{p} > \int_{\partial z_2(Q)} R(\mathbf{p}) d\mathbf{p}. \quad (2.1)$$

Since Q is open, $\partial z_1(Q)$ and $\partial z_2(Q)$ are Lebesgue measurable in \mathbb{R}^d , due to [3, Property (D)]. If we let $\partial z_1(Q)$ be the unit open ball in \mathbb{R}^d , and $\partial z_2(Q)$ be $\partial z_1(Q)$ minus any $(d-1)$ -dimensional subset P' satisfying $\overline{P'} \subset \partial z_1(Q)$. For any $\mathbf{p} \in \partial P'$, then \mathbf{p} is contained in the boundary of $\partial z_2(Q)$ and $\mathbf{p} \in \text{Int}(\partial z_1(Q))$. But it is easy to see that (2.1) does not hold for this case. On the other hand, due to [3, Property (B)], $\partial z_1(\overline{Q})$ and $\partial z_2(\overline{Q})$ are closed. Thus, if there is $\mathbf{p} \in \mathbb{R}^d$ such that \mathbf{p} is contained in the boundary of $\partial z_2(\overline{Q})$, $\mathbf{p} \in \text{Int}(\partial z_1(\overline{Q}))$, and $\partial z_2(\overline{Q}) \subset \partial z_1(\overline{Q})$, then

$$\int_{\partial z_1(\overline{Q})} R(\mathbf{p}) d\mathbf{p} > \int_{\partial z_2(\overline{Q})} R(\mathbf{p}) d\mathbf{p}. \quad (2.2)$$

Thus in our strategy, to get Lemma 2.3, a revised form of [3, Lemma 10.2], we shall prove (2.2) instead of (2.1).

Secondly, our Lemma 2.3 is significantly different from [3, Lemma 10.2] that we only consider a point $\mathbf{x}_0 \in \partial \overline{Q}$ such that z_2 is differentiable at \mathbf{x}_0 . Then $\partial z_2(\mathbf{x}_0)$ is just a single point in \mathbb{R}^d , which would simplify the analysis a lot.

The proof of Theorem 2.1 would depend on the following two lemmas:

Lemma 2.2. *Let $z \in W^+(\Omega)$ and Q be an open subset of Ω with $\overline{Q} \subset \Omega$. We assume that T is a hyperplane whose equation is*

$$z = z_T + \mathbf{p}_T \cdot (\mathbf{x} - \mathbf{x}_T),$$

where $\mathbf{p}_T, \mathbf{x}_T \in \mathbb{R}^d$, and $z_T \in \mathbb{R}$. In addition, we suppose

$$z(\mathbf{x}) \geq z_T + \mathbf{p}_T \cdot (\mathbf{x} - \mathbf{x}_T), \quad \forall \mathbf{x} \in \overline{Q}, \quad (2.3a)$$

$$\exists \mathbf{x}_0 \in \partial \overline{Q} \text{ such that } (\mathbf{x}_0, z(\mathbf{x}_0)) \in T \text{ and } \mathbf{p}_T \notin \partial z(\mathbf{x}_0). \quad (2.3b)$$

Then, $\mathbf{p}_T \notin \partial z(\overline{Q})$.

Proof. We prove it by contradiction. Assume $\mathbf{p}_T \in \partial z(\overline{Q})$, then $\exists \mathbf{x}_Q \in \overline{Q}$ such that it holds:

$$z(\mathbf{x}) \geq z(\mathbf{x}_Q) + \mathbf{p}_T \cdot (\mathbf{x} - \mathbf{x}_Q), \quad \forall \mathbf{x} \in \Omega.$$

By (2.3b), we know that $\exists \mathbf{x}_1 \in \Omega$ such that

$$z(\mathbf{x}_1) < z(\mathbf{x}_0) + \mathbf{p}_T \cdot (\mathbf{x}_1 - \mathbf{x}_0).$$

Combining the two estimates above, we infer

$$z(\mathbf{x}_Q) + \mathbf{p}_T \cdot (\mathbf{x}_1 - \mathbf{x}_Q) \leq z(\mathbf{x}_1) < z(\mathbf{x}_0) + \mathbf{p}_T \cdot (\mathbf{x}_1 - \mathbf{x}_0)$$

which indicates that

$$z(\mathbf{x}_0) > z(\mathbf{x}_Q) + \mathbf{p}_T \cdot (\mathbf{x}_0 - \mathbf{x}_Q).$$

Apply (2.3a) to the latest inequality, we arrive at $z(\mathbf{x}_0) > z_T + \mathbf{p}_T \cdot (\mathbf{x}_0 - \mathbf{x}_T)$, which contradicts with the fact: $(\mathbf{x}_0, z(\mathbf{x}_0)) \in T$ from (2.3b). \square

The following Lemma 2.3 is a revised version of [3, Lemma 10.2]:

Lemma 2.3. *Let $z_1, z_2 \in W^+(\Omega)$ and Q be any open subset of Ω such that the following conditions hold: $\overline{Q} \subset \Omega$, $z_1 < z_2$ in Q and $z_1 = z_2$ on ∂Q . Assume that for any $\mathbf{x}_Q \in \partial \overline{Q}$, $\exists r > 0$ such that $\overline{B_r(\mathbf{x}_Q)} \subset \Omega$ and $z_1 \geq z_2$ in $B_r(\mathbf{x}_Q) \setminus \overline{Q}$.*

If there exists some point $\mathbf{x}_0 \in \partial \overline{Q}$ such that z_2 is differentiable at \mathbf{x}_0 and $\nabla z_2(\mathbf{x}_0) \notin \partial z_1(\mathbf{x}_0)$, then it holds:

$$\int_{\partial z_1(\overline{Q})} R(\mathbf{p}) d\mathbf{p} > \int_{\partial z_2(\overline{Q})} R(\mathbf{p}) d\mathbf{p}.$$

Proof. By the assumptions it is easy to see that $\partial z_2(Q) \subset \partial z_1(Q)$.

For any $\mathbf{x}_Q \in \partial \overline{Q}$, let T' be a supporting hyperplane of the graph of z_2 at $(\mathbf{x}_Q, z_2(\mathbf{x}_Q))$. Then the equation of T' is $y = z_2(\mathbf{x}_Q) + \mathbf{p}_{T'} \cdot (\mathbf{x} - \mathbf{x}_Q)$ with some $\mathbf{p}_{T'} \in \partial z_2(\mathbf{x}_Q)$. Thus $\exists r > 0$ such that the following holds:

$$z_1(\mathbf{x}) \geq z_2(\mathbf{x}) \geq L_{T'}(\mathbf{x}) := z_2(\mathbf{x}_Q) + \mathbf{p}_{T'} \cdot (\mathbf{x} - \mathbf{x}_Q), \quad \forall \mathbf{x} \in B_r(\mathbf{x}_Q) \setminus \overline{Q}.$$

If $\mathbf{p}_{T'} \notin \partial z_1(\mathbf{x}_Q)$, then by the above inequality and the condition that $z_1 = z_2$ on ∂Q , there exists $\mathbf{x}_1 \in Q$ such that $z_1(\mathbf{x}_1) < L_{T'}(\mathbf{x}_1)$. Thus there exists a new hyperplane parallel to T' such that it supports the graph of z_1 at some point $(\mathbf{x}_2, z_1(\mathbf{x}_2))$ with $\mathbf{x}_2 \in Q$. Hence $\mathbf{p}_{T'} \in \partial z_1(Q)$. Furthermore we claim that

$$\mathbf{p}_{T'} \in \text{Int}(\partial z_1(Q)).$$

In fact, if we choose $\epsilon > 0$ and $\mathbf{p}_\epsilon \in B_\epsilon(\mathbf{p}_{T'}) \subset \mathbb{R}^d$, then $L_{T'}^\epsilon := z_1(\mathbf{x}_Q) + \mathbf{p}_\epsilon \cdot (\mathbf{x} - \mathbf{x}_Q)$ satisfies the following:

$$L_{T'}^\epsilon(\mathbf{x}) \geq L_{T'}(\mathbf{x}) - \epsilon \left(\sup_{\mathbf{x}, \mathbf{x}' \in Q} |\mathbf{x} - \mathbf{x}'| \right), \quad \forall \mathbf{x} \in Q.$$

Then $L_{T'}^\epsilon(\mathbf{x}_1) > z_1(\mathbf{x}_1)$ for ϵ small enough. Thus there is a supporting hyperplane of the graph of z_1 at some point $(\mathbf{x}_\epsilon, z_1(\mathbf{x}_\epsilon))$ with $\mathbf{x}_\epsilon \in Q$, and its equation is $y = z_1(\mathbf{x}_\epsilon) + \mathbf{p}_\epsilon \cdot (\mathbf{x} - \mathbf{x}_\epsilon)$. Therefore $\mathbf{p}_\epsilon \in \partial z_1(Q)$ for ϵ small enough, which concludes the claim.

Hence for any $\mathbf{x}_Q \in \partial \overline{Q}$, it holds: $\partial z_2(\mathbf{x}_Q) \setminus \partial z_1(\mathbf{x}_Q) \subset \text{Int}(\partial z_1(Q))$ and $\partial z_2(\overline{Q}) \subset \partial z_1(\overline{Q})$ since $\partial z_2(Q) \subset \partial z_1(Q)$.

In the following, let T_i be a supporting hyperplane of the graph of z_i at $(\mathbf{x}_0, z_i(\mathbf{x}_0))$ and the equations of T_i be

$$z = z_i(\mathbf{x}_0) + \mathbf{p}_i \cdot (\mathbf{x} - \mathbf{x}_0),$$

for $i = 1, 2$, $\mathbf{p}_1 \in \partial z_1(\mathbf{x}_0)$ and $\mathbf{p}_2 = \nabla z_2(\mathbf{x}_0)$. For any $0 < \lambda < 1$, let T_λ be a hyperplane given by the equation: $z = z_1(\mathbf{x}_0) + \mathbf{p}_\lambda \cdot (\mathbf{x} - \mathbf{x}_0)$, where $\mathbf{p}_\lambda := (1 - \lambda)\mathbf{p}_2 + \lambda\mathbf{p}_1$. Obviously, $(\mathbf{x}_0, z_1(\mathbf{x}_0)) = (\mathbf{x}_0, z_2(\mathbf{x}_0)) \in T_\lambda$, and $z_2(\mathbf{x}) \geq z_2(\mathbf{x}_0) + \mathbf{p}_\lambda \cdot (\mathbf{x} - \mathbf{x}_0)$, $\forall \mathbf{x} \in \overline{Q}$ and any $0 < \lambda < 1$. Since z_2 is differentiable at \mathbf{x}_0 , then we claim that

$$\mathbf{p}_\lambda \notin \partial z_2(\mathbf{x}_0), \quad \forall 0 < \lambda < 1.$$

In fact, if not, then $\exists \lambda_0 \in (0, 1)$ such that $\mathbf{p}_{\lambda_0} = \nabla z_2(\mathbf{x}_0)$, which implies that $\mathbf{p}_1 = \mathbf{p}_2$. Then we arrive at a contradiction.

Since $\mathbf{p}_2 = \lim_{\lambda \rightarrow 0+} \mathbf{p}_\lambda$ and $\mathbf{p}_\lambda \notin \partial z_2(\mathbf{x}_0)$, $\forall 0 < \lambda < 1$, then \mathbf{p}_2 belongs to the boundary of $\partial z_2(\overline{Q})$. Applying the argument before for $\mathbf{p}_{T'}$ to the fact: $\mathbf{p}_2 = \nabla z_2(\mathbf{x}_0) \notin \partial z_1(\mathbf{x}_0)$, we get $\mathbf{p}_2 \in \text{Int}(\partial z_1(Q))$. By [3, Property B in Section 9.4], $\partial z_2(\overline{Q})$ is compact in \mathbb{R}^d . Then we have

$$\int_{\partial z_1(\overline{Q})} R(\mathbf{p}) d\mathbf{p} > \int_{\partial z_2(\overline{Q})} R(\mathbf{p}) d\mathbf{p}.$$

□

Finally we go to the proof of Theorem 2.1:

Proof of Theorem 2.1. We prove it by contradiction. Assume $\{\mathbf{x} \in \Omega : z_1(\mathbf{x}) < z_2(\mathbf{x})\} \neq \emptyset$ and Q is a connected component of $\{\mathbf{x} \in \Omega : z_1(\mathbf{x}) < z_2(\mathbf{x})\}$. We define $\epsilon_0 := \sup_{\mathbf{x} \in Q} (z_2(\mathbf{x}) - z_1(\mathbf{x}))$, $z_2^{(1)} := z_2(\mathbf{x}) - \epsilon_0/2$ and $Q^{(1)} := \{\mathbf{x} \in Q : z_1(\mathbf{x}) < z_2^{(1)}(\mathbf{x})\}$. Obviously, $\epsilon_0 > 0$ and $Q^{(1)} \neq \emptyset$. Without the loss of generality, we assume that $Q^{(1)}$ is connected. From the assumptions on z_1 and z_2 , we know that $\overline{Q^{(1)}} \subset \Omega$. Then by Lemma 2.3, for any $\mathbf{x} \in \partial \overline{Q^{(1)}}$ such that $z_2^{(1)}$ is differentiable at \mathbf{x} , it holds: $\nabla z_2(\mathbf{x}) = \nabla z_2^{(1)}(\mathbf{x}) \in \partial z_1(\mathbf{x})$.

In the following, we claim, any $\mathbf{x}_0 \in Q^{(1)}$ such that z_2 is differentiable at \mathbf{x}_0 , that

$$\nabla z_2(\mathbf{x}_0) \in \partial z_1(\mathbf{x}_0). \quad (2.4)$$

If (2.4) is not true, we define $\epsilon_1 := z_2^{(1)}(\mathbf{x}_0) - z_1(\mathbf{x}_0)$, $z_2^{(2)} := z_2^{(1)} - \epsilon_1$, $Q^{(2)} := \{\mathbf{x} \in Q^{(1)} : z_1(\mathbf{x}) < z_2^{(2)}(\mathbf{x})\}$. Obviously, $z_2^{(1)}(\mathbf{x}_0) > z_2^{(2)}(\mathbf{x}_0) = z_1(\mathbf{x}_0)$ and we infer

- (1) $Q^{(2)} \neq \emptyset$. In fact, if not, then $z_1 \geq z_2^{(2)}$ in $Q^{(1)}$. Since $z_1(\mathbf{x}_0) = z_2^{(2)}(\mathbf{x}_0)$, then $\nabla z_2(\mathbf{x}_0) = \nabla z_2^{(2)}(\mathbf{x}_0) \in \partial z_1(\mathbf{x}_0)$, which contradicts with our assumption.
- (2) $\mathbf{x}_0 \in \overline{Q^{(2)}}$. If not, then $\exists r_1 > 0$ such that $\overline{B_{r_1}(\mathbf{x}_0)} \subset Q^{(1)}$ and $B_{r_1}(\mathbf{x}_0) \cap Q^{(2)} = \emptyset$. Hence $z_1 \geq z_2^{(2)}$ on $\overline{B_{r_1}(\mathbf{x}_0)}$. Since $z_1(\mathbf{x}_0) = z_2^{(2)}(\mathbf{x}_0)$, then it holds: $\nabla z_2(\mathbf{x}_0) \in \partial z_1(\mathbf{x}_0)$, which is a contradiction.
- (3) $\mathbf{x}_0 \in \partial \overline{Q^{(2)}}$. If not, then by (2), we know that $\mathbf{x}_0 \in \text{Int}(\overline{Q^{(2)}})$. That is, $\exists r_2 > 0$ such that $B_{r_2}(\mathbf{x}_0) \subset \text{Int}(\overline{Q^{(2)}})$ and $\overline{B_{r_2}(\mathbf{x}_0)} \subset Q^{(1)}$. Hence $z_1 \leq z_2^{(2)}$ in $B_{r_2}(\mathbf{x}_0)$. Since $z_1(\mathbf{x}_0) = z_2^{(2)}(\mathbf{x}_0)$, we get $\partial z_1(\mathbf{x}_0) = \{\nabla z_2(\mathbf{x}_0)\}$, which is contradiction.

Since $\mathbf{x}_0 \in \overline{\partial Q^{(2)}}$, there exists $r_3 > 0$ such that $\overline{B_{r_3}(\mathbf{x}_0)} \subset Q^{(1)}$ and $z_1 \geq z_2^{(2)}$ in $B_{r_3}(\mathbf{x}_0) \setminus Q^{(2)}$. By Lemma 2.3, it holds:

$$\int_{\partial z_1(Q^{(2)})} R(\mathbf{p}) d\mathbf{p} > \int_{\partial z_2(Q^{(2)})} R(\mathbf{p}) d\mathbf{p},$$

which arrives at a contradiction.

Since $z_1, z_2, z_2^{(1)}, z_2^{(2)} \in W^+(\Omega)$, then they are all differentiable almost everywhere in Ω . By (2.4), we know that $\nabla(z_1 - z_2^{(1)})(\mathbf{x}) = 0$ for a. e. $\mathbf{x} \in Q^{(1)}$, which, together with [20, Corollary 2.1.9] and the fact: $z_1 = z_2^{(1)}$ on $\partial Q^{(1)}$ to (2.5), implies that $z_1 = z_2^{(1)}$ in $\overline{Q^{(1)}}$. This is a contraction with our assumption. \square

3. THE BORDER OF A CONVEX FUNCTION

In this part, the boundary behaviour of convex functions would be considered. Firstly, we give the definition of border of convex functions on convex domains, which was firstly introduced by Bakelman[3, Section 10.4].

Definition 3.1. (The border of a convex function) For any $v \in W^+(\Omega)$, we define the border of v to be a function on $\partial\Omega$ by

$$b_v(\mathbf{x}_0) = \liminf_{\mathbf{x} \rightarrow \mathbf{x}_0} v(\mathbf{x}), \quad \forall \mathbf{x}_0 \in \partial\Omega.$$

The following Lemma 3.2 shows that if $b_v \in C^0(\partial\Omega)$, then v can be extended continuously to $\overline{\Omega}$ such that $v|_{\partial\Omega} = b_v$.

Lemma 3.2. Let $v \in W^+(\Omega)$ and b_v be the border of v as in Definition 3.1. For any $\mathbf{x}_0 \in \partial\Omega$, if b_v is continuous at \mathbf{x}_0 , then we have

$$b_v(\mathbf{x}_0) = \lim_{\mathbf{x} \rightarrow \mathbf{x}_0} v(\mathbf{x}).$$

Proof. We choose an orthogonal coordinate of \mathbb{R}^d such that in an open neighborhood of \mathbf{x}^0 , $\partial\Omega$ can be represented as

$$(y^1, \dots, y^{d-1}, z(y^1, \dots, y^{d-1})) \text{ satisfying } z(y^1, \dots, y^{d-1}) \geq 0.$$

For simplicity, \mathbf{x}^0 is taken to be the origin and for any $\mathbf{x} \in \mathbb{R}^d$, we denote by (x^1, \dots, x^d) its coordinate. We define

$$\tilde{b}_v(\mathbf{x}) := \limsup_{\mathbf{y} \rightarrow \mathbf{x}} v(\mathbf{y}), \quad \forall \mathbf{x} \in \partial\Omega.$$

To prove Lemma 3.2, it is sufficient to show that $\tilde{b}_v(\mathbf{x}_0) = b_v(\mathbf{x}_0)$ and we prove it by contradiction. Assume $\epsilon := \tilde{b}_v(\mathbf{x}_0) - b_v(\mathbf{x}_0) > 0$. By the continuity of b_v at \mathbf{x}_0 , there is $\delta_1 > 0$ such that the following holds:

$$|b_v(y^1, \dots, y^{d-1}, z(y^1, \dots, y^{d-1})) - b_v(\mathbf{x}_0)| < \epsilon/3, \quad \forall (y^1, \dots, y^{d-1}) \in B_{2\delta_1}(\mathbf{0}) \subset \mathbb{R}^{d-1}.$$

We define

$$S := \{\mathbf{x} \in \Omega : |(x^1, \dots, x^{d-1})| < 2\delta_1 \text{ and } \exists (y^1, \dots, y^{d-1}) \in \overline{B_{\delta_1}(\mathbf{0})} \subset \mathbb{R}^{d-1} \\ \text{such that it holds: } |b_v(y^1, \dots, y^{d-1}, z(y^1, \dots, y^{d-1})) - v(\mathbf{x})| < \epsilon/3\}.$$

Then we claim that for any $\delta_2 > 0$ and any $(y^1, \dots, y^{d-1}) \in \overline{B_{\delta_1}(\mathbf{0})} \subset \mathbb{R}^{d-1}$, there exists $\mathbf{x} \in S$ such that

$$|(y^1, \dots, y^{d-1}, z(y^1, \dots, y^{d-1})) - \mathbf{x}| < \delta_2. \quad (3.1)$$

In fact, if (3.1) is not true, then there exist $0 < \bar{\delta}_2 \leq \delta_1$ and $(\bar{y}^1, \dots, \bar{y}^{d-1}) \in \overline{B_{\delta_1}(\mathbf{0})} \subset \mathbb{R}^{d-1}$, such that for any $\mathbf{x} \in S$, we infer

$$|(\bar{y}^1, \dots, \bar{y}^{d-1}, z(\bar{y}^1, \dots, \bar{y}^{d-1})) - \mathbf{x}| \geq \bar{\delta}_2 > 0.$$

By the definition of S , for any $\mathbf{x} \in \Omega$ with $|(\bar{y}^1, \dots, \bar{y}^{d-1}, z(\bar{y}^1, \dots, \bar{y}^{d-1})) - \mathbf{x}| < \bar{\delta}_2$, we know that $\mathbf{x} \notin S$ and

$$|b_v(\bar{y}^1, \dots, \bar{y}^{d-1}, z(\bar{y}^1, \dots, \bar{y}^{d-1})) - v(\mathbf{x})| \geq \epsilon/3 > 0,$$

which contradicts with the definition of b_v . Thus the claim holds true.

It is easy to see that there exist d points $\{(y_i^1, \dots, y_i^{d-1})\}_{i=1}^d$ such that $|(y_i^1, \dots, y_i^{d-1})| = \delta_1$ for all $1 \leq i \leq d$ and

$$\sum_{i=1}^d d^{-1}(y_i^1, \dots, y_i^{d-1}) = \mathbf{0} \in \mathbb{R}^{d-1}.$$

By (3.1), there exist a constant $0 < \sigma < 1$ and $\{\mathbf{x}_i\}_{i=1}^d$ in S , such that

$$\overline{B_{\sigma\delta_1}(\mathbf{0})} \subset \mathbb{R}^{d-1} \text{ is contained in the convex hull of } \{(x_i^1, \dots, x_i^{d-1})\}_{i=1}^d \text{ in } \mathbb{R}^{d-1}. \quad (3.2)$$

Let T be the hyperplane in \mathbb{R}^d passing through $\{\mathbf{x}_i\}_{i=1}^d$. Then the equation of T is $x^d = a^1 x^1 + \dots + a^{d-1} x^{d-1} + c$. Since $x_i^d > 0$ for any $1 \leq i \leq d$, then $c > 0$. Thus from the definition of \tilde{b}_v , there is $\bar{\mathbf{x}} \in \Omega$ such that

$$(\bar{x}^1, \dots, \bar{x}^{d-1}) \in B_{\frac{1}{2}\sigma\delta_1}(\mathbf{0}) \subset \mathbb{R}^{d-1}, \quad (3.3a)$$

$$|\tilde{b}_v(\mathbf{x}_0) - v(\bar{\mathbf{x}})| < \epsilon/3, \quad (3.3b)$$

$$\bar{x}^d < a^1 \bar{x}^1 + \dots + a^{d-1} \bar{x}^{d-1} + c. \quad (3.3c)$$

Here (3.3c) holds true since $\bar{\mathbf{x}}$ can be chosen as close to $\mathbf{x}_0 = \mathbf{0}$ as we need. By (3.2, 3.3a), there are $0 < \mu_i < 1$ for any $1 \leq i \leq d$ such that $\mu_1 + \dots + \mu_d = 1$ and

$$(\bar{x}^1, \dots, \bar{x}^{d-1}) = \sum_{i=1}^d \mu_i (x_i^1, \dots, x_i^{d-1}), \quad \sum_{i=1}^d \mu_i (x_i^1, \dots, x_i^{d-1}, x_i^d) \in T.$$

By the fact that $z(\bar{x}^1, \dots, \bar{x}^{d-1}) < \bar{x}^d$ and (3.3c), there is $0 < \lambda < 1$ such that

$$\bar{x}^d = (1 - \lambda)z(\bar{x}^1, \dots, \bar{x}^{d-1}) + \lambda \sum_{i=1}^d \mu_i x_i^d,$$

and $(\bar{x}^1, \dots, \bar{x}^{d-1}, z(\bar{x}^1, \dots, \bar{x}^{d-1})), \mathbf{x}_1, \dots, \mathbf{x}_d$ are in a general location in \mathbb{R}^d . Hence we get

$$\bar{\mathbf{x}} = (1 - \lambda)(\bar{x}^1, \dots, \bar{x}^{d-1}, z(\bar{x}^1, \dots, \bar{x}^{d-1})) + \lambda \sum_{i=1}^d \mu_i (\bar{x}_i^1, \dots, \bar{x}_i^{d-1}, \bar{x}_i^d),$$

and $\bar{\mathbf{x}}$ is contained in the interior of the convex hull of $(\bar{x}^1, \dots, \bar{x}^{d-1}, z(\bar{x}^1, \dots, \bar{x}^{d-1})) \cup \{\mathbf{x}_i\}_{i=1}^d$. Due to (3.1), we can take $\mathbf{x}_{d+1} \in S$ close enough to $(\bar{x}^1, \dots, \bar{x}^{d-1}, z(\bar{x}^1, \dots, \bar{x}^{d-1}))$, such that the point $\bar{\mathbf{x}}$ is contained in the convex hull of $\{\mathbf{x}_i\}_{i=1}^{d+1}$. Then we can infer

$$v(\bar{\mathbf{x}}) \leq \max_{1 \leq i \leq d+1} v(\bar{x}_i).$$

By the construction of the set S , $v(\mathbf{x}) < b_v(\mathbf{x}_0) + 2\epsilon/3$ for any $\mathbf{x} \in S$. Thus we have

$$v(\bar{\mathbf{x}}) < b_v(\mathbf{x}_0) + 2\epsilon/3 = \tilde{b}_v(\mathbf{x}_0) - \epsilon/3.$$

This contradicts with (3.3b). □

4. CONVERGENCE OF A SEQUENCE OF CONVEX FUNCTIONS

Throughout this section, we denote by $\{\Omega_n\}_{n=1}^{+\infty}$ a sequence of open convex subdomains of Ω , and $\{v_n\}_{n=1}^{+\infty}$ a sequence of convex functions with

$$v_n \in W^+(\Omega_n), \quad \forall n \in \mathbb{N}. \quad (4.1)$$

Furthermore, we assume that for any $\delta > 0$, there is $N = N(\delta) \in \mathbb{N}$,

$$\overline{\Omega_\delta} \subset \Omega_n \subset \Omega, \quad \forall n \geq N. \quad (4.2)$$

This section would consist of the following two parts:

4.1. Convergence of a sequence of convex functions inside domain.

The main result of this subsection is Theorem 4.1.

Theorem 4.1. *We assume that (4.1), (4.2) hold and there is $M < +\infty$ such that it holds:*

$$\|v_n\|_{L^\infty(\Omega_n)} \leq M, \quad \forall n \in \mathbb{N}. \quad (4.3)$$

Then there is a subsequence $\{v_{n_k}\}_{k=1}^{+\infty}$ of $\{v_n\}_{n=1}^{+\infty}$ and a function $v_0 \in W^+(\Omega)$, such that for any $\delta > 0$,

$$\|v_{n_k} - v_0\|_{L^\infty(\overline{\Omega_\delta})} \longrightarrow 0 \quad \text{as } k \rightarrow +\infty.$$

Moreover, if we define the set functions ν_k and ν_0 by

$$\begin{aligned} \nu_k(e) &:= \int_{\partial v_{n_k}(e)} R(\mathbf{p}) d\mathbf{p}, \quad \forall \text{ Borel set } e \subset \Omega_{n_k}, \\ \nu_0(e) &:= \int_{\partial v_0(e)} R(\mathbf{p}) d\mathbf{p}, \quad \forall \text{ Borel set } e \subset \Omega. \end{aligned}$$

Then, ν_0 is a measure in Ω , and ν_k is a measure in Ω_{n_k} for any $n \in \mathbb{N}$. Furthermore, it holds: $\nu_k \rightharpoonup \nu_0$ weakly, as $k \rightarrow +\infty$, i.e. for any $f \in C_c(\Omega)$, it holds:

$$\int_{\Omega_{n_k}} f d\nu_k \rightarrow \int_{\Omega} f d\nu_0, \quad \text{as } k \rightarrow +\infty.$$

The proof of Theorem 4.1 comes from Lemma 4.2 and Lemma 4.3 immediately.

Lemma 4.2. *We assume that (4.1), (4.2) and (4.3) hold. Then there is a subsequence $\{v_{n_k}\}_{k=1}^{+\infty}$ of $\{v_n\}_{n=1}^{+\infty}$ and a function $v_0 \in W^+(\Omega)$, such that for any $\delta > 0$,*

$$\|v_{n_k} - v_0\|_{L^\infty(\overline{\Omega_\delta})} \longrightarrow 0 \quad \text{as } k \rightarrow +\infty.$$

Proof. By (4.2), we know that there exists some $N = N(\delta) \in \mathbb{N}$ such that $\overline{\Omega_\delta} \subset \Omega_n$ and $\text{dist}(\partial\Omega_n, \Omega_\delta) \geq \delta/2$ for all $n \geq N$. By (4.3), it is easy to see that

$$\sup_{\mathbf{x}, \mathbf{y} \in \overline{\Omega_\delta}} |v_n(\mathbf{x}) - v_n(\mathbf{y})| \leq \varrho_\delta \cdot \delta/2, \quad \forall n \geq N.$$

Here $\varrho_\delta := 4M/\delta$. By the convexity of $\{v_n\}_{n=1}^{+\infty}$, we infer that $\partial v_n(\overline{\Omega_\delta}) \subset \overline{B_{\varrho_\delta}(\mathbf{0})} \subset \mathbb{R}^d$ for all $n \geq N$. Therefore for any $\mathbf{p} \in \cap_{n \geq N} \partial v_n(\overline{\Omega_\delta})$, we have that $|\mathbf{p}| \leq \varrho_\delta$ and

$$v_n(\mathbf{x}) - v_n(\mathbf{y}) \geq \mathbf{p} \cdot (\mathbf{x} - \mathbf{y}) \geq -\varrho_\delta \cdot |\mathbf{x} - \mathbf{y}|, \quad \forall \mathbf{x}, \mathbf{y} \in \overline{\Omega_\delta}.$$

This statement implies the equicontinuity of $\{v_n\}_{n \geq N}$ on $\overline{\Omega_\delta}$.

Therefore, by Ascoli-Arzelà Theorem, there exists a function $v_0 \in W^+(\Omega)$ and a subsequence $\{v_{n_k}\}_{k=1}^{+\infty}$ of $\{v_n\}_{n=1}^{+\infty}$, such that for any $\delta > 0$

$$\lim_{k \rightarrow +\infty} \|v_{n_k} - v_0\|_{L^\infty(\overline{\Omega_\delta})} = 0.$$

□

Lemma 4.3. *Let (4.1, 4.2) hold. We assume that there is a function $v_0 \in W^+(\Omega)$, such that for any $\delta > 0$,*

$$\lim_{n \rightarrow +\infty} \|v_n - v_0\|_{L^\infty(\overline{\Omega_\delta})} = 0. \quad (4.4)$$

We define the set functions ν_n and ν_0 by

$$\begin{aligned} \nu_n(e) &:= \int_{\partial v_n(e)} R(\mathbf{p}) d\mathbf{p}, \quad \forall \text{ Borel set } e \subset \Omega_n, \\ \nu_0(e) &:= \int_{\partial v_0(e)} R(\mathbf{p}) d\mathbf{p}, \quad \forall \text{ Borel set } e \subset \Omega. \end{aligned}$$

Then, ν_0 is a measure in Ω , and ν_n is a measure in Ω_n for any $n \in \mathbb{N}$. Furthermore, for any $f \in C_c^0(\Omega)$,

$$\int_{\Omega_n} f d\nu_n \rightarrow \int_{\Omega} f d\nu_0, \quad \text{as } n \rightarrow +\infty.$$

Proof. Since $R > 0$ and $R \in L^1_{loc}(\mathbb{R}^d)$, thus ν_0 is a measure in Ω , and ν_n is a measure in Ω_n for any $n \in \mathbb{N}$ (see [11, Theorem 1.1.13]).

By (4.2), for any compact set $F \subset \Omega$ and any open set Q with $\overline{Q} \subset \Omega$, we have that $F \subset \Omega_n$ and $\overline{Q} \subset \Omega_n$ for any $n \in \mathbb{N}$ large enough. By (4.4) and [11, Lemma 1.2.2], there hold: $\limsup_{n \rightarrow +\infty} \partial v_n(F) \subset \partial v_0(F)$ and $\liminf_{n \rightarrow +\infty} \partial v_n(Q) \supset \partial v_0(Q)$ for any compact set $K \subset Q$. Then by Fatou Lemma, we obtain

$$\limsup_{n \rightarrow +\infty} \nu_n(F) \leq \nu_0(F) \quad \text{and} \quad \liminf_{n \rightarrow +\infty} \nu_n(Q) \geq \nu_0(Q), \quad (4.5)$$

which implies that

$$\lim_{n \rightarrow +\infty} \nu_n(B) = \nu_0(B). \quad (4.6)$$

for any Borel set $B \subset \Omega$ with $\overline{B} \subset \Omega$ and $\nu_0(\partial B) = 0$. Now we choose $f \in C_c^0(\Omega)$ with $f \geq 0$. (in fact, we can write $f = f^+ - f^-$). By (4.2), we have that for $n \in \mathbb{N}$ large enough,

$$\int_{\Omega_n} f d\nu_n = \int_0^{+\infty} \nu_n(\{\mathbf{x} \in \Omega_n : f(\mathbf{x}) > t\}) dt = \int_0^{+\infty} \nu_n(\{\mathbf{x} \in \Omega : f(\mathbf{x}) > t\}) dt.$$

Let $B_t := \{\mathbf{x} \in \Omega : f(\mathbf{x}) > t\}, \forall t > 0$. Since $f \in C_c^0(\Omega)$, then B_t is Borel and $\partial B_t \subset A_t := \{\mathbf{x} \in \Omega : f(\mathbf{x}) = t\}$ for any $t > 0$. Furthermore, if $n \in \mathbb{N}$ large enough, $B_t \subset \text{Supp}(f) \subset \Omega_n$. By foliations of Borel sets (see [16, Proposition 2.16]), we know that $\nu_0(A_t) > 0$ for at most countably many $t \in (0, +\infty)$. Hence, $\exists J \subset (0, +\infty)$ with $|J| > 0$ and $|(0, +\infty) \setminus J| = 0$ such that $\nu_0(A_t) = 0, \forall t \in J$. This implies that $\nu_0(\partial B_t) = 0, \forall t \in J$. From (4.6), we obtain

$$\lim_{n \rightarrow +\infty} \nu_n(B_t) = \nu_0(B_t), \quad \forall t \in J. \quad (4.7)$$

Moreover by (4.5), there is a positive constant C_0 such that for any $n \in \mathbb{N}$

$$\nu_n(B_t) \leq C_0(1 + \nu_0(\text{Supp}(f))) \cdot \chi_{[0, \overline{f}]}(t) \quad (4.8)$$

where $\overline{f} := \sup_{\mathbf{x} \in \Omega} f(\mathbf{x})$ and $\chi_{[0, \overline{f}]}$ denotes the characteristic function on $[0, \overline{f}]$. By dominated convergence Theorem, we infer, from (4.7) and (4.8), that

$$\int_{\Omega_n} f d\nu_n = \int_0^{+\infty} \nu_n(B_t) dt \rightarrow \int_0^{+\infty} \nu_0(B_t) dt = \int_{\Omega} f d\nu_0, \text{ as } n \rightarrow +\infty.$$

□

4.2. Convergence of a sequence of borders of convex functions.

In this subsection, we study the convergence property of border of a sequence of convex functions. The main results in this part are Lemma 4.8 and Theorem 4.10.

Before giving main results, we firstly need to introduce some important notations (see also Bakelman[3]). Let \mathbf{a}_0 be any point of $\partial\Omega$. Then there is a supporting $(d-1)$ -plane α of $\partial\Omega$ passing through \mathbf{a}_0 , an open d -ball $U_\rho(\mathbf{a}_0)$ with the center \mathbf{a}_0 , and the radius $\rho > 0$ such that the convex $(d-1)$ -surface

$$\Gamma_\rho(\mathbf{a}_0) := \partial\Omega \cap U_\rho(\mathbf{a}_0)$$

has the one-to-one orthogonal projection $\Pi_\alpha : \Gamma_\rho(\mathbf{a}_0) \rightarrow \alpha$. Moreover, the unit normal of α in the direction of the halfspace of \mathbb{R}^d , where $\overline{\Omega}$ stays, passes through interior points of Ω . Let $x^1, \dots, x^{d-1}, x^d, z$ be the Cartesian coordinates in \mathbb{R}^{d+1} with the following properties:

- \mathbf{a}_0 is the origin.
- The axes x^1, \dots, x^{d-1} stay in the plane α .
- The axis x^d is directed along the interior normal of $\partial\Omega$ at the point \mathbf{a}_0 .
- The axis z is orthogonal to \mathbb{R}^d .

Clearly, the convex $(d-1)$ -surface $\Gamma_\rho(\mathbf{a}_0)$ is the graph of $g(x^1, \dots, x^{d-1}) \in W^+(\Pi_\alpha(\Gamma_\rho(\mathbf{a}_0)))$. Obviously, $g(0, \dots, 0) = 0$ and $g(x^1, \dots, x^{d-1}) \geq 0$ for all points of the set $\Pi_\alpha(\Gamma_\rho(\mathbf{a}_0))$.

Definition 4.4. (Local parabolic support) We shall say that $\partial\Omega$ has a local parabolic support of order $\tau \geq 0$ at the point \mathbf{a}_0 if there are positive numbers ρ_0 and $\eta(\mathbf{a}_0)$ such that

$$g(x^1, \dots, x^{d-1}) \geq \eta(\mathbf{a}_0)(|x^1|^2 + \dots + |x^{d-1}|^2)^{\frac{\tau+2}{2}}, \quad \forall (x^1, \dots, x^{d-1}) \in \Pi_\alpha(\Gamma_\rho(\mathbf{a}_0)).$$

Definition 4.5. (Boundary having a parabolic support) We shall say that $\partial\Omega$ has a parabolic support of order not smaller than a constant $0 \leq \tau < +\infty$, if the local parabolic support of $\partial\Omega$ has order not smaller than τ at all points $\mathbf{a} \in \partial\Omega$.

Definition 4.6. (Topological limit of sets[3, Section 3.4]) Let $\{E_n\}_{n=1}^{+\infty}$ be a sequence of subsets in \mathbb{R}^d and we denote by $\overline{\lim}_{n \rightarrow +\infty}^T E_n$ the superior topological limit of $\{E_n\}_{n=1}^{+\infty}$, which is defined as

$$\mathbf{x} \in \overline{\lim}_{n \rightarrow +\infty}^T E_n \Leftrightarrow \exists \text{ a subsequence } \{n_k\}_{k=1}^\infty \text{ and } \mathbf{x}_{n_k} \in E_{n_k} \text{ such that } \lim_{k \rightarrow +\infty} \mathbf{x}_{n_k} = \mathbf{x}.$$

We also denote by $\underline{\lim}_{n \rightarrow +\infty}^T E_n$ the inferior topological limit of $\{E_n\}_{n=1}^{+\infty}$, which is defined as

$$\mathbf{x} \in \underline{\lim}_{n \rightarrow +\infty}^T E_n \Leftrightarrow \exists \mathbf{x}_n \in E_n \text{ such that } \lim_{n \rightarrow +\infty} \mathbf{x}_n = \mathbf{x}.$$

If $\overline{\lim}_{n \rightarrow +\infty}^T E_n = \underline{\lim}_{n \rightarrow +\infty}^T E_n$, then we say $\{E_n\}_{n=1}^{+\infty}$ has a topological limit, written as $\lim_{n \rightarrow +\infty}^T E_n$, which is equal to $\overline{\lim}_{n \rightarrow +\infty}^T E_n$ or $\underline{\lim}_{n \rightarrow +\infty}^T E_n$.

Assumption 4.1. $\partial\Omega$ has a parabolic support (see Definition 4.5) of order not smaller than a constant $0 \leq \tau < +\infty$.

Remark 4.7. if Assumption 4.1 holds for $\partial\Omega$, then the domain Ω is strictly convex.

Assumption 4.2. $R \in L^1_{loc}(\mathbb{R}^d)$ and the following holds:

$$R(\mathbf{p}) \geq C_0 |\mathbf{p}|^{-2k}, \quad \forall \mathbf{p} \in \mathbb{R}^d \text{ with } |\mathbf{p}| \geq r_0 > 0.$$

Here, $k \geq 0$, $C_0 > 0$ and $r_0 > 0$ are some constants.

Assumption 4.3. Let $\{\Omega_n\}_{n=1}^{+\infty}$ be a sequence of open convex subdomains of Ω satisfying (4.2), and $\{v_n\}_{n=1}^{+\infty}$ a sequence of convex functions satisfying (4.1). We assume that the following conditions are fulfilled:

- (a) There is a function $v_0 \in W^+(\Omega)$ such that $\lim_{n \rightarrow +\infty} v_n(\mathbf{x}) = v_0(\mathbf{x})$, $\forall \mathbf{x} \in \Omega$.
- (b) There exist two uniform constants $C_1 > 0$ and $\lambda \geq 0$ such that for any $\mathbf{x}_0 \in \partial\Omega$, there exists an open d -ball $U_\rho(\mathbf{x}_0)$ such that

$$\liminf_{n \rightarrow +\infty} \int_{\partial v_n(e \cap \Omega_n)} R(\mathbf{p}) d\mathbf{p} \leq C_1 \left(\sup_{\mathbf{x} \in e} \text{dist}(\mathbf{x}, \partial\Omega) \right)^\lambda |e|, \quad \forall \text{ Borel set } e \subset U_\rho(\mathbf{x}_0) \cap \Omega.$$

Assumption 4.4. Let $\{\Omega_n\}_{n=1}^{+\infty}$ be a sequence of open convex subdomains of Ω satisfying (4.2), and $\{v_n\}_{n=1}^{+\infty}$ be a sequence of convex functions satisfying (4.1). Let b_n be the border of v_n for any $n \in \mathbb{N}$. Let \mathbf{S}_n be the graphs of b_n for any $n \in \mathbb{N}$. We assume that

- (a) $b_n \in C^0(\partial\Omega_n)$ for any $n \in \mathbb{N}$.
- (b) There is $\tilde{b} \in C^0(\partial\Omega)$ such that $\lim_{n \rightarrow +\infty}^T \mathbf{S}_n = \tilde{\mathbf{S}}$, where $\tilde{\mathbf{S}}$ is the graph of \tilde{b} .

The following Lemma 4.8 gives some important convergent property for the borders of a sequence of convex functions:

Lemma 4.8. *Let $\{\Omega_n\}_{n=1}^{+\infty}$ be a sequence of open convex subdomains of Ω satisfying (4.2), and $\{v_n\}_{n=1}^{+\infty}$ a sequence of convex functions satisfying (4.1). Let part (a) of Assumption 4.3 and Assumption 4.4 hold. Let b_0 be the border of v_0 introduced in Assumption 4.3. Then*

$$b_0(\mathbf{x}) \leq \tilde{b}(\mathbf{x}), \quad \forall \mathbf{x} \in \partial\Omega.$$

Here, \tilde{b} is a function on $\partial\Omega$ introduced in Assumption 4.4.

Remark 4.9. In the proof [3, Theorem 10.6], it has been stated that the conclusion of Lemma 4.8 is trivial. However, we have found that it is really not trivial and the proof needs rather tricky analysis.

Proof. We shall prove it by contradiction. By part (a) of Assumption 4.4 and Lemma 3.2, we can extend v_n to $\overline{\Omega_n}$ such that

$$v_n \in C^0(\overline{\Omega_n}), \quad v_n|_{\partial\Omega_n} = b_n, \quad \forall n \in \mathbb{N}.$$

Fixing an $\mathbf{x} \in \partial\Omega$, then we can take $\mathbf{x}' \in \partial\Omega \setminus \{\mathbf{x}\}$ such that the interior of $\overline{\mathbf{x}\mathbf{x}'}$ is contained in Ω , where $\overline{\mathbf{x}\mathbf{x}'}$ denotes the line segment between \mathbf{x} and \mathbf{x}' . By (4.2), for any $n \in \mathbb{N}$ large enough, there are $\mathbf{x}_n, \mathbf{x}'_n \in \partial\Omega_n$, such that

$$\overline{\mathbf{x}_n\mathbf{x}'_n} \subset \overline{\mathbf{x}\mathbf{x}'}, \quad \overline{\mathbf{x}_n\mathbf{x}'_n} \subset \overline{\Omega_n}, \quad \mathbf{x}_n \neq \mathbf{x}'_n.$$

We choose an orthogonal coordinate $\{x, x^2, \dots, x^d\}$ such that $\overline{\mathbf{x}\mathbf{x}'}$ and $\{\overline{\mathbf{x}_n\mathbf{x}'_n}\}$ are all contained in $\{(y, y^2, \dots, y^d) \in \mathbb{R}^d : y^2 = \dots = y^d = 0\}$. Hence, we can use $x, x' \in \mathbb{R}$ to represent \mathbf{x} and \mathbf{x}' respectively and use $x_n, x'_n \in \mathbb{R}$ to represent \mathbf{x}_n and \mathbf{x}'_n respectively for any $n \in \mathbb{N}$ large enough. Without loss of generality, we assume that

$$x < x_n < x'_n < x', \quad \forall n \in \mathbb{N} \text{ large enough.}$$

Due to (4.2), it is easy to see that $x_n \rightarrow x$ as $n \rightarrow +\infty$.

We assume $b_0(x) > \tilde{b}(x)$ and define $\epsilon := b_0(x) - \tilde{b}(x)$. In the following we claim that

$$b_n(x_n) \longrightarrow \tilde{b}(x), \quad \text{as } n \rightarrow +\infty. \quad (4.9)$$

In fact, $(x_n, b_n(x_n)) \in \mathcal{S}_n$ for all $n \in \mathbb{N}$. Let $\{x_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{x_n\}_{n=1}^{\infty}$, such that $\lim_{k \rightarrow +\infty} (x_{n_k}, b_{n_k}(x_{n_k})) = (x, \liminf_{n \rightarrow +\infty} b_n(x_n))$. Due to part (b) of Assumption 4.4 and Definition 4.6, one obtains that $(x, \liminf_{n \rightarrow +\infty} b_n(x_n)) \in \tilde{\mathcal{S}}$, where $\tilde{\mathcal{S}}$ is the graph of $\tilde{b} \in C^0(\partial\Omega)$.

Similarly, we can show that $(x, \limsup_{n \rightarrow +\infty} b_n(x_n)) \in \tilde{\mathcal{S}}$. Then we get $\tilde{b}(x) = \liminf_{n \rightarrow +\infty} b_n(x_n) = \limsup_{n \rightarrow +\infty} b_n(x_n)$. Therefore (4.9) is true.

Due to the definition of b_0 , for any $\delta > 0$, there exists some $x'' \in (x, x + \delta)$ such that

$$|v_0(x'') - b_0(x)| < \epsilon/6.$$

According to (4.2), part (a) of Assumption 4.3 and (4.9), there is $x_{n_\delta} \in (x, x'') \subset (x, x + \delta)$ such that there hold:

$$|v_{n_\delta}(x'') - v_0(x'')| < \epsilon/6 \text{ and } |b_{n_\delta}(x_{n_\delta}) - \tilde{b}(x)| < \epsilon/6.$$

Since $v_n \in C^0(\overline{\Omega_n})$ and $v_n|_{\partial\Omega_n} = b_n$ for any $n \in \mathbb{N}$, then there exists some $x''_{n_\delta} \in (x_{n_\delta}, x'')$ such that

$$|v_{n_\delta}(x''_{n_\delta}) - b_{n_\delta}(x_{n_\delta})| < \epsilon/6.$$

By the latest three estimates above, one obtains that

$$|v_{n_\delta}(x'') - b_0(x)| < \epsilon/3 \quad \text{and} \quad |v_{n_\delta}(x''_{n_\delta}) - \tilde{b}(x)| < \epsilon/3. \quad (4.10)$$

Taking $\tilde{x} := (x + x')/2$, from (4.2), we know that $\tilde{x} \in \Omega_n$ for all $n \in \mathbb{N}$ large enough and $x''_{n_\delta} < x'' < \tilde{x}$ for $\delta > 0$ small enough. By the convexity of v_{n_δ} , (4.10) and the definition of ϵ , we have

$$\begin{aligned} (x'' - x''_{n_\delta})v_{n_\delta}(\tilde{x}) &\geq (\tilde{x} - x''_{n_\delta})v_{n_\delta}(x'') - (\tilde{x} - x'')v_{n_\delta}(x''_{n_\delta}) \\ &\geq (b_0(x) - \epsilon/3)(\tilde{x} - x''_{n_\delta}) - (\tilde{b}(x) + \epsilon/3)(\tilde{x} - x'') \\ &= (b_0(x) - \epsilon/3)(x'' - x''_{n_\delta}) + \epsilon(\tilde{x} - x'')/3, \end{aligned}$$

which, together with the constructions of x'' , x''_{n_δ} and \tilde{x} , implies that

$$\frac{\tilde{x} - x''}{x'' - x''_{n_\delta}} \longrightarrow +\infty \quad \text{as } \delta \rightarrow 0.$$

This leads to $v_{n_\delta}(\tilde{x}) \rightarrow +\infty$ as $\delta \rightarrow 0$, a contradiction with part (a) of Assumption 4.3. \square

The following Theorem 4.10 is a minor revision of [3, Theorem 10.6]. The proof of [3, Theorem 10.6] consists of three parts. Its first part is very geometrically intuitive. We rewrite the proof with more detailed explanation in Appendix A.

Theorem 4.10. *Let $\{\Omega_n\}_{n=1}^{+\infty}$ be a sequence of open convex subdomains of Ω satisfying (4.2), and $\{v_n\}_{n=1}^{+\infty}$ a sequence of convex functions satisfying (4.1). Let Assumptions 4.1, 4.2, 4.3, 4.4 hold for Ω , the function R , and $\{v_n\}_{n=1}^{+\infty}$. Now let the numbers k, λ and τ satisfy:*

$$\begin{cases} k \leq K & \text{if } 0 \leq k < 1 \text{ or } k \geq \frac{d}{2}, \\ k < K & \text{if } 1 \leq k < \frac{d}{2} \end{cases}$$

where $K = \frac{d + \tau + 1}{\tau + 2} + \frac{\lambda}{2}$. Let b_0 be the border of v_0 introduced in Assumption 4.3 and \tilde{b} be the function on $\partial\Omega$ introduced in Assumption 4.4. Then

$$\tilde{b}(\mathbf{x}) = b_0(\mathbf{x}), \quad \forall \mathbf{x} \in \partial\Omega.$$

5. THE FINITE ELEMENT METHOD

In this section, we first introduce the concept of mesh, which is a sequence of convex polyhedra domains with standard triangulation to approximate the convex domain Ω . Then, we design a finite element method to approximate the exact solution of (1.1) and we show that this finite element method is well-posed.

5.1. The mesh.

In this part, we firstly give definition of the mesh, which plays an important role in the finite element method. Furthermore, we show that the convex domain Ω can be approximated by a sequence of convex polyhedra domains.

Definition 5.1. For a given positive real number h , we denote by \mathcal{T}_h a set of d -dimensional simplexes contained in $\overline{\Omega}$ such that the following conditions (5.1a) -(5.1d) are fulfilled:

$$\begin{aligned} & \text{for any } T, T' \in \mathcal{T}_h \text{ with } T \neq T', T \cap T' \text{ is a } \bar{d}\text{-dimensional sub-simplex of} \\ & \text{both } T \text{ and } T'. \text{ Here, } 0 \leq \bar{d} \leq d-1; \end{aligned} \quad (5.1a)$$

$$h = \max_{T \in \mathcal{T}_h} h_T, \text{ where } h_T \text{ is the diameter of } T \in \mathcal{T}_h; \quad (5.1b)$$

$$\Omega_h := \text{Int}(\overline{\cup_{T \in \mathcal{T}_h} T}) \text{ is a convex domain}; \quad (5.1c)$$

$$\text{all vertexes of } \partial\Omega_h \text{ are contained on } \partial\Omega. \quad (5.1d)$$

Then \mathcal{T}_h is called a mesh of Ω .

Lemma 5.2. For any $\delta > 0$, there is a polyhedra domain P_δ such that $\overline{\Omega_\delta} \subset P_\delta \subset \overline{P_\delta} \subset \overline{\Omega}$, and all vertexes of P_δ are contained on $\partial\Omega$. Here $\Omega_\delta := \{\mathbf{x} \in \Omega : \text{dist}(\mathbf{x}, \partial\Omega) > \delta\}$ for any $\delta > 0$.

Proof. $\forall \epsilon > 0$, we define

$$C_\epsilon := \{(i_1\epsilon, (i_1+1)\epsilon] \times \cdots \times (i_d\epsilon, (i_d+1)\epsilon] : \forall i_1, \dots, i_d \in \mathbb{Z}\}.$$

Obviously, \mathbb{R}^d can be covered by cubes in C_ϵ without overlapping. For $0 < \epsilon \leq \frac{1}{3\sqrt{d}}\delta$, there exist finitely many $\{\mathbf{C}_i\}_{i=1}^m \subset C_\epsilon$ such that $\mathbf{C}_i \cap \partial\Omega \neq \emptyset$ for any $i = 1, \dots, m$. Taking any point, denoted by \mathbf{B}_i , in $\mathbf{C}_i \cap \partial\Omega$ for any $1 \leq i \leq m$, we define P_δ to be the convex hull of $\{\mathbf{B}_i\}_{i=1}^m$. Then all vertexes of P_δ stay on $\partial\Omega$ and $\overline{P_\delta} \subset \overline{\Omega}$ since Ω is convex.

In the following, we shall show that $\overline{\Omega_\delta} \subset P_\delta$. In fact, if not, then there is a point $\mathbf{x}_0 \in \overline{\Omega_\delta}$ and $\mathbf{x}_0 \notin P_\delta$. Without loss of generality, we may assume that

$$\mathbf{x}_0 = (0, \dots, 0) \in \mathbb{R}^d \text{ and } x^d < 0 \text{ for any } (x^1, \dots, x^d) = \mathbf{x} \in P_\delta.$$

Since $\overline{\Omega_\delta} \subset \Omega$, there is $x^d > 0$ such that the point $\mathbf{x}' := (0, \dots, 0, x^d) \in \partial\Omega$. Due to the definition of Ω_δ , $\text{dist}(\mathbf{x}_0, \mathbf{x}') = x^d \geq \delta$. Obviously, there is a cube $\mathbf{C}' \in C_\epsilon$ such that $\mathbf{x}' \in \mathbf{C}'$. Then there is some positive integer j with $1 \leq j \leq m$ such that $\mathbf{B}_j \in \mathbf{C}'$ and $\text{dist}(\mathbf{x}', \mathbf{B}_j) \leq \sqrt{d}\epsilon \leq \delta/3$, which implies that

$$x_j^d \geq 2\delta/3 > 0 \text{ where } (x_j^1, \dots, x_j^d) = \mathbf{B}_j.$$

This contradicts with the fact that $\mathbf{B}_j \in P_\delta$. Therefore $\overline{\Omega_\delta} \subset P_\delta$. \square

According to Lemma 5.2 and standard triangulation for polyhedra, there is $I \subset (0, 1)$ such that the following conditions are fulfilled:

$$0 \text{ is the unique accumulation point of } I; \quad (5.2a)$$

$$\text{for any } h \in I, \text{ there is a mesh } \mathcal{T}_h \text{ of } \Omega; \quad (5.2b)$$

$$\text{for any } \delta > 0, \text{ there is } h_\delta > 0 \text{ such that } \overline{\Omega_\delta} \subset \Omega_h \text{ if } h \in I \text{ and } h < h_\delta. \quad (5.2c)$$

In fact, the proof of Lemma 5.2 is constructive such that it naturally provides an algorithm to construct the convex polyhedra to approximate Ω .

5.2. The finite element method.

For any given mesh \mathcal{T}_h of Ω , we denote the vertexes of \mathcal{T}_h contained in the interior of Ω_h and the vertexes of $\partial\Omega_h$ by $\{\mathbf{A}_i\}_{i=1}^{k_h}$ and $\{\mathbf{B}_j\}_{j=1}^{m_h}$, respectively, and we define $M_h(z_1, \dots, z_{k_h})$ to be the convex hull of $\{(\mathbf{A}_i, z_i)\}_{i=1}^{k_h} \cup \{(\mathbf{B}_j, g(\mathbf{B}_j))\}_{j=1}^{m_h}$ in \mathbb{R}^{d+1} for any real numbers $\{z_i\}_{i=1}^{k_h}$. We introduce $K_h := \{M_h(z_1, \dots, z_{k_h}) : \forall z_i \in \mathbb{R}, 1 \leq i \leq k_h\}$ and

$$H_h := \{v \in W^+(\Omega_h) \cap C(\overline{\Omega_h}) : \exists M_h \in K_h \text{ such that } v(\mathbf{x}) = \inf_{(\mathbf{x}, z) \in M_h} z, \forall \mathbf{x} \in \overline{\Omega_h}\}. \quad (5.3)$$

The so-called finite element method is to find $u_h \in H_h$ such that

$$\int_{\partial u_h(\mathbf{A}_i)} R(\mathbf{p}) d\mathbf{p} = \int_{\Omega_h} \phi_{i,h} d\mu, \quad \forall 1 \leq i \leq k_h, \quad (5.4)$$

where, for any $1 \leq i \leq k_h$, $\phi_{i,h} \in C_c(\Omega_h) \cap \mathbb{P}_1(\mathcal{T}_h)$ with the conditions:

$$\phi_{i,h}(\mathbf{A}_j) = \delta_{ij}, \quad \forall 1 \leq j \leq k_h \quad (5.5)$$

and $\mathbb{P}_1(\mathcal{T}_h)$ is defined to be the set of piecewise linear functions on \mathcal{T}_h .

Definition 5.3. A domain $\Omega \subset \mathbb{R}^d$ is called strictly convex if for any $\mathbf{x}, \mathbf{x}' \in \overline{\Omega}$,

$$\lambda \mathbf{x} + (1 - \lambda) \mathbf{x}' \in \Omega, \quad \forall 0 < \lambda < 1.$$

Lemma 5.4. Assume that Ω is strictly convex and \mathcal{T}_h is a mesh of Ω . Let $\{\mathbf{A}_i\}_{i=1}^{k_h}$ and $\{\mathbf{B}_j\}_{j=1}^{m_h}$ are the vertexes of \mathcal{T}_h contained in the interior of Ω_h and the vertexes of $\partial\Omega_h$, respectively. Then there hold:

$$H_h \neq \emptyset; \quad (5.6a)$$

$$v(\mathbf{B}_j) = g(\mathbf{B}_j), \quad \forall v \in H_h \text{ and } 1 \leq j \leq m_h; \quad (5.6b)$$

$$w = v \text{ on } \partial\Omega_h, \quad \forall w, v \in H_h. \quad (5.6c)$$

Proof. For any $1 \leq i \leq k_h$, we take

$$z_i = \left(\max_{1 \leq j \leq m_h} g(\mathbf{B}_j) \right) + 1.$$

Then $M_h := M_h(z_1, \dots, z_{k_h})$ is the convex hull of $\{\mathbf{B}_j\}_{j=1}^{m_h}$. We define

$$v(\mathbf{x}) = \inf_{(\mathbf{x}, z) \in M_h} z, \quad \forall \mathbf{x} \in \overline{\Omega_h}.$$

Obviously, $v \in W^+(\Omega_h) \cap C(\overline{\Omega_h})$. Thus $v \in H_h$, which shows that $H_h \neq \emptyset$.

Since Ω is strictly convex and (5.1d) holds true for any $1 \leq j \leq m_h$, \mathbf{B}_j is not contained in the convex hull of $\{(\mathbf{A}_i, z_i)\}_{i=1}^{k_h} \cup \{(\mathbf{B}_l, g(\mathbf{B}_l))\}_{l=1, l \neq j}^{m_h}$. Then we obtain (5.6b). Finally from (5.6b) and (5.3), the statement (5.6c) holds true. \square

Assumption 5.1. We assume that

$$\int_{\Omega} d\mu < \int_{\mathbb{R}^d} R(\mathbf{p}) d\mathbf{p}.$$

Theorem 5.5. Let \mathcal{T}_h be a mesh of Ω . We assume that Ω is strictly convex and Assumption 5.1 holds. Then, the finite element method (5.4) has a unique solution.

Proof. By Assumption 5.1, we know that

$$\sum_{i=1}^{k_h} \int_{\Omega_h} \phi_{i,h} d\mu = \int_{\Omega_h} \left(\sum_{i=1}^{k_h} \phi_{i,h} \right) d\mu \leq \int_{\Omega_h} d\mu \leq \int_{\Omega} d\mu < \int_{\mathbb{R}^d} R(\mathbf{p}) d\mathbf{p}. \quad (5.7)$$

Now we replace the set H in the proof of [3, Theorem 11.1] by

$$\{v \in H_h : \int_{\partial v(\mathbf{A}_i)} R(\mathbf{p}) d\mathbf{p} \leq \int_{\Omega_h} \phi_{i,h} d\mu, \quad \forall 1 \leq i \leq k_h\},$$

where H_h is introduced in (5.3). By (5.6, 5.7), the proof of [3, Theorem 11.1] can go through, such that we can conclude that the finite element method (5.4) has a unique solution. \square

6. CONVERGENCE OF THE FINITE ELEMENT METHOD TO (1.1)

In this section, we show that under suitable assumptions, (1.1) is well-posed and the solutions of the finite element method (5.4) converges to the exact solution. Theorem 6.5 is the main result.

6.1. Convergence of border of solutions of the finite element method.

Before we prove the convergence of solutions of the finite element method (5.4), we firstly give Lemma 6.1 and 6.2, which show the convergence of border of finite element solutions.

Lemma 6.1. *Let $I \subset (0, 1)$ satisfy (5.2a, 5.2b, 5.2c) and Σ_h be the set of all $(d-1)$ -dimensional closed polyhedra on $\partial\Omega_h$. If Ω is strictly convex, then*

$$\lim_{I \ni h \rightarrow 0} \sup_{\mathbf{K} \in \Sigma_h} \left(\sup_{\mathbf{x}, \mathbf{x}' \in \mathbf{K}} |\mathbf{x} - \mathbf{x}'| \right) = 0.$$

Proof. We prove it by contradiction. If Lemma 6.1 does not hold true, then there is $\{h_n\}_{n=1}^{+\infty} \subset I$ such that $\lim_{n \rightarrow +\infty} h_n = 0$ and for any $n \in \mathbb{N}$, there exists some $\mathbf{K}_n \in \Sigma_{h_n}$ such that the following condition holds true:

$$\sup_{\mathbf{x}, \mathbf{x}' \in \mathbf{K}_n} |\mathbf{x} - \mathbf{x}'| \geq \epsilon_0$$

for some positive constant ϵ_0 . For any $n \in \mathbb{N}$, since \mathbf{K}_n is compact, then there are two vertexes $\mathbf{x}'_n, \mathbf{x}''_n$ of \mathbf{K}_n such that

$$|\mathbf{x}'_n - \mathbf{x}''_n| = \sup_{\mathbf{x}, \mathbf{x}' \in \mathbf{K}_n} |\mathbf{x} - \mathbf{x}'| \geq \epsilon_0.$$

By (5.1d), $\mathbf{x}'_n, \mathbf{x}''_n \in \partial\Omega$, for any $n \in \mathbb{N}$. Without loss of generality (we always can take a subsequence of $\{h_n\}_{n=1}^{+\infty}$ if necessary), we have that

$$\lim_{n \rightarrow +\infty} \mathbf{x}'_n = \bar{\mathbf{x}}' \in \partial\Omega \quad \text{and} \quad \lim_{n \rightarrow +\infty} \mathbf{x}''_n = \bar{\mathbf{x}}'' \in \partial\Omega.$$

Then by the latest two estimates above, we can see that $|\bar{\mathbf{x}}' - \bar{\mathbf{x}}''| \geq \epsilon_0 > 0$. Since Ω is strictly convex, then $\lambda \bar{\mathbf{x}}' + (1 - \lambda) \bar{\mathbf{x}}'' \in \Omega$, $\forall 0 < \lambda < 1$. By the definition of Ω_δ , we can choose $\delta > 0$ small enough such that $\lambda \bar{\mathbf{x}}' + (1 - \lambda) \bar{\mathbf{x}}'' \in \text{Int}(\Omega_\delta)$, for all $1/3 < \lambda < 2/3$. Then, by (5.2c), we get that $(\mathbf{x}'_n + \mathbf{x}''_n)/2 \in \text{Int}(\Omega_\delta) \subset \Omega_{h_n}$ if n is large enough, which arrives at a contradiction since $\mathbf{x}'_n, \mathbf{x}''_n$ are two vertexes of \mathbf{K}_n and $\mathbf{K}_n \in \Sigma_h \subset \partial\Omega_h$ is convex. \square

Lemma 6.2. *Let $I \subset (0, 1)$ satisfy (5.2a, 5.2b, 5.2c) and Ω be strictly convex. We define*

$$\mathbf{S}_0 := \{(\mathbf{x}, g(\mathbf{x})) : \forall \mathbf{x} \in \partial\Omega\} \text{ and } \mathbf{S}_h := \{(\mathbf{x}, v(\mathbf{x})) : \forall \mathbf{x} \in \partial\Omega_h\}, \forall v \in H_h.$$

Then \mathbf{S}_h is independent of the choice of $v \in H_h$ and it is a $(d-1)$ -dimensional surface homeomorphic to the $(d-1)$ -unit sphere. Furthermore, we have that $\lim_{I \ni h \rightarrow 0}^T \mathbf{S}_h = \mathbf{S}_0$.

Proof. According to (5.6c), it is easy to see that \mathbf{S}_h is independent of the choice of $v \in H_h$ and it is a $(d-1)$ -dimensional surface homeomorphic to the $(d-1)$ -unit sphere.

In the following, we prove that $\lim_{I \ni h \rightarrow 0}^T \mathbf{S}_h = \mathbf{S}_0$. Firstly we define a function $g_h : \partial\Omega_h \rightarrow \mathbb{R}$ by $(\mathbf{x}, g_h(\mathbf{x})) \in \mathbf{S}_h$, $\forall \mathbf{x} \in \partial\Omega_h$. We take $\mathbf{x}_0 \in \partial\Omega$ arbitrarily, and for any $h \in I$, we define \mathbf{x}_h to be a vertex on $\partial\Omega_h$ which reaches the shortest distance between \mathbf{x}_0 and all vertexes $(\{\mathbf{B}_j\}_{j=1}^{m_h})$ on $\partial\Omega_h$. Obviously, $\lim_{I \ni h \rightarrow 0} \mathbf{x}_h = \mathbf{x}_0$. Since $g \in C(\partial\Omega)$, $\lim_{I \ni h \rightarrow 0} (\mathbf{x}_h, g(\mathbf{x}_h)) = (\mathbf{x}, g(\mathbf{x}))$, which implies that

$$\mathbf{S}_0 \subset \underline{\lim}_{I \ni h \rightarrow 0}^T \mathbf{S}_h.$$

So, it is sufficient to show that

$$\overline{\lim}_{I \ni h \rightarrow 0}^T \mathbf{S}_h \subset \mathbf{S}_0.$$

We take $\{h_n\}_{n=1}^{+\infty} \subset I$ such that $\lim_{n \rightarrow +\infty} h_n = 0$ and choose $\{\mathbf{x}_n\}_{n=1}^{+\infty} \subset \partial\Omega_{h_n}$ such that $\lim_{n \rightarrow +\infty} (\mathbf{x}_n, g_{h_n}(\mathbf{x}_n))$ exists. By (5.2c), we know that $\lim_{n \rightarrow +\infty} \mathbf{x}_n \in \overline{\Omega} \setminus \Omega$. Thus we obtain that

$$\lim_{n \rightarrow +\infty} (\mathbf{x}_n, g_{h_n}(\mathbf{x}_n)) = (\mathbf{x}_0, z_0) \quad \text{where } \mathbf{x}_0 \in \partial\Omega \text{ and } z_0 \in \mathbb{R}.$$

In the following, we show that $z_0 = g(\mathbf{x}_0)$. $\forall n \in \mathbb{N}$, there is a $(d-1)$ -dimensional closed polyhedra \mathbf{K}_n such that $\mathbf{x}_n \in \mathbf{K}_n \subset \partial\Omega_{h_n}$. We denote by $\{\mathbf{B}_{i,n}\}_{i=1}^{l_n}$ all vertexes of \mathbf{K}_n (l_n may not have a uniform bound). For \mathbf{K}_n , since Lemma 6.1 holds and $\mathbf{x}_n \rightarrow \mathbf{x}_0$ as $n \rightarrow +\infty$, we have that

$$\lim_{n \rightarrow +\infty} \sup_{\mathbf{x}, \mathbf{x}' \in \mathbf{K}_n} |\mathbf{x} - \mathbf{x}'| = \lim_{n \rightarrow +\infty} \text{dist}(\mathbf{x}_0, \mathbf{K}_n) = 0$$

which, together with (5.6b) and $g \in C(\partial\Omega)$, implies that

$$\max_{1 \leq i \leq l_n} |g_{h_n}(\mathbf{B}_{i,n}) - g(\mathbf{x}_0)| \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Since $\mathbf{x}_n \in \mathbf{K}_n$, there are nonnegative numbers $\{\lambda_{i,n}\}_{1 \leq i \leq l_n}$ such that there hold:

$$\begin{cases} \lambda_{1,n} + \cdots + \lambda_{l_n,n} = 1, \\ \lambda_{1,n} \mathbf{B}_{1,n} + \cdots + \lambda_{l_n,n} \mathbf{B}_{l_n,n} = \mathbf{x}_n, \\ \lambda_{1,n} g_{h_n}(\mathbf{B}_{1,n}) + \cdots + \lambda_{l_n,n} g_{h_n}(\mathbf{B}_{l_n,n}) = g_{h_n}(\mathbf{x}_n). \end{cases}$$

Here the third equality follows from the definitions of \mathbf{K}_n , \mathbf{S}_h and g_h . Then, it holds:

$$\begin{aligned} |g_{h_n}(\mathbf{x}_n) - g(\mathbf{x}_0)| &\leq \sum_{i=1}^{l_n} \lambda_{i,n} |g_{h_n}(\mathbf{B}_{i,n}) - g(\mathbf{x}_0)| \\ &\leq \max_{1 \leq i \leq l_n} |g_{h_n}(\mathbf{B}_{i,n}) - g(\mathbf{x}_0)| \rightarrow 0 \quad \text{as } n \rightarrow +\infty. \end{aligned}$$

Therefore, $z_0 = g(\mathbf{x}_0)$. □

6.2. Convergence of finite element method.

Lemma 6.3. *Assume Ω is strictly convex. If Assumption 5.1 holds and $I \subset (0, 1)$ satisfies (5.2a, 5.2b, 5.2c), then there exists a uniform constant $M > 0$ such that for any $h \in I$, the solution of the finite element method (5.4), denoted by u_h , satisfies :*

$$\|u_h\|_{L^\infty(\Omega_h)} \leq M, \quad \forall h \in I.$$

Proof. By the construction of $\{u_h\}_{h \in I}$ in (5.4) and Lemma 5.4, it holds for any $n \in \mathbb{N}$,

$$\min_{\mathbf{y} \in \partial\Omega} g(\mathbf{y}) \leq u_h(\mathbf{x}) \leq \max_{\mathbf{y} \in \partial\Omega} g(\mathbf{y}), \quad \forall \mathbf{x} \in \partial\Omega_h.$$

Since $u_h \in W^+(\Omega_h)$, then we can derive that

$$u_h(\mathbf{x}) \leq \max_{\mathbf{y} \in \partial\Omega} g(\mathbf{y}), \quad \forall \mathbf{x} \in \Omega_h.$$

In the following, we will deduce some lower bound for u_h in Ω_h . Let $\mathbf{x}_h \in \Omega_h$ such that

$$u_h(\mathbf{x}_h) = \min_{\mathbf{x} \in \Omega_h} u_h(\mathbf{x}).$$

Without loss of generality, we assume $u_h(\mathbf{x}_h) < \min_{\mathbf{y} \in \partial\Omega} g(\mathbf{y})$. We define

$$\rho_h := \frac{\min_{\mathbf{y} \in \partial\Omega} g(\mathbf{y}) - u_h(\mathbf{x}_h)}{\sup_{\mathbf{x}, \mathbf{x}' \in \Omega} |\mathbf{x} - \mathbf{x}'|}.$$

Then it is easy to see that $\overline{B_{\rho_h}(\mathbf{0})} \subset \partial u_h(\Omega_h) \subset \mathbb{R}^d$, which implies that

$$\int_{B_{\rho_h}(\mathbf{0})} R(\mathbf{p}) d\mathbf{p} \leq \int_{\partial u_h(\Omega_h)} R(\mathbf{p}) d\mathbf{p}.$$

By the construction of u_h and Assumption 5.1, we know that

$$\int_{\partial u_h(\Omega_h)} R(\mathbf{p}) d\mathbf{p} = \int_{\Omega_h} \left(\sum_{i=1}^{k_h} \phi_{i,h} \right) d\mu \leq \int_{\Omega_h} d\mu \leq \int_{\Omega} d\mu < \int_{\mathbb{R}^d} R(\mathbf{p}) d\mathbf{p}.$$

Then by combining the latest two estimates above, it holds:

$$\int_{B_{\rho_h}(\mathbf{0})} R(\mathbf{p}) d\mathbf{p} \leq \int_{\Omega} d\mu < \int_{\mathbb{R}^d} R(\mathbf{p}) d\mathbf{p}.$$

We set $g_R(\rho) := \int_{B_\rho(\mathbf{0})} R(\mathbf{p}) d\mathbf{p}$ for all $\rho > 0$ and $\omega_0 := \int_{\Omega} d\mu$. Obviously, $g_R : [0, +\infty) \rightarrow [0, +\infty)$ is strictly increasing and g_R^{-1} exists (it is also strictly increasing and continuous). Then we infer that

$$g_R(\rho_h) \leq \omega_0 < \int_{\mathbb{R}^d} R(\mathbf{p}) d\mathbf{p},$$

which implies that $0 < \rho_h \leq g_R^{-1}(\omega_0) < +\infty$. Hence by the definition of g_h , we get

$$u_h(\mathbf{x}_h) \geq \min_{\mathbf{y} \in \partial\Omega} g(\mathbf{y}) - \left(\sup_{\mathbf{x}, \mathbf{x}' \in \Omega} |\mathbf{x} - \mathbf{x}'| \right) \cdot g_R^{-1}(\omega_0).$$

Therefore, for any $h \in I$, it holds:

$$\min_{\mathbf{y} \in \partial\Omega} g(\mathbf{y}) - \left(\sup_{\mathbf{x}, \mathbf{x}' \in \Omega} |\mathbf{x} - \mathbf{x}'| \right) \cdot g_R^{-1}(\omega_0) \leq u_h(\mathbf{x}) \leq \max_{\mathbf{y} \in \partial\Omega} g(\mathbf{y}), \quad \forall \mathbf{x} \in \Omega_h.$$

□

Assumption 6.1. For any $\mathbf{x}_0 \in \partial\Omega$, there exists an open d -ball $U_\rho(\mathbf{x}_0)$ such that

$$\int_{e \cap \Omega} d\mu \leq C'_1 \left(\sup_{\mathbf{x} \in e} \text{dist}(\mathbf{x}, \partial\Omega) \right)^\lambda |e|, \quad \forall \text{ Borel set } e \subset U_\rho(\mathbf{x}_0) \cap \Omega.$$

Here, $C'_1 > 0$ and $\lambda \geq 0$ are constants independent of the choice of $\mathbf{x}_0 \in \partial\Omega$.

Lemma 6.4. *Let Assumptions 4.1, 4.2, 5.1, 6.1 hold and the numbers k, λ and τ satisfy*

$$\begin{cases} k \leq K & \text{if } 0 \leq k < 1 \text{ or } k \geq d/2, \\ k < K & \text{if } 1 \leq k < d/2, \end{cases}$$

where $K = \frac{d + \tau + 1}{\tau + 2} + \frac{\lambda}{2}$. If $I \subset (0, 1)$ satisfies (5.2a, 5.2b, 5.2c) and u_h is the solution of the finite element method (5.4) for any $h \in I$, then there exist a sequence $\{h_n\}_{n=1}^{+\infty} \subset I$ with $\lim_{n \rightarrow +\infty} h_n = 0$, and $u_0 \in W^+(\Omega) \cap C(\overline{\Omega})$ such that u_0 solves (1.1) and

$$\lim_{n \rightarrow +\infty} \|u_{h_n} - u_0\|_{L^\infty(\overline{\Omega_\delta})} = 0, \quad \forall \delta > 0,$$

where $\Omega_\delta = \{\mathbf{x} \in \Omega : \text{dist}(\mathbf{x}, \partial\Omega) > \delta\}$.

Proof. According to (5.2c) and Lemma 6.3, we can apply Theorem 4.1 to $\{u_h\}_{h \in I}$. Therefore, by Theorem 4.1, there exist a sequence $\{h_n\}_{n=1}^{+\infty} \subset I$ with $\lim_{n \rightarrow +\infty} h_n = 0$, and a function $u_0 \in W^+(\Omega)$ such that

$$\begin{cases} \lim_{n \rightarrow +\infty} \|u_{h_n} - u_0\|_{L^\infty(\overline{\Omega_\delta})} = 0, & \forall \delta > 0, \\ \lim_{n \rightarrow +\infty} \int_{\Omega_{h_n}} f d\mu_n = \int_{\Omega} f d\mu_0, & \forall f \in C_c^0(\Omega), \end{cases}$$

where μ_n, μ_0 are measures in Ω_{h_n}, Ω defined as

$$\begin{cases} \mu_n(e) = \int_{\partial u_{h_n}(e)} R(\mathbf{p}) d\mathbf{p}, & \forall \text{ Borel set } e \subset \Omega_{h_n}, \\ \mu_0(e) = \int_{\partial u_0(e)} R(\mathbf{p}) d\mathbf{p}, & \forall \text{ Borel set } e \subset \Omega. \end{cases}$$

From the construction of u_h in (5.4), we know that for any $f \in C_c(\Omega)$,

$$\int_{\Omega_{h_n}} f d\mu_n = \sum_{i=1}^{k_{h_n}} f(\mathbf{A}_i) \int_{\Omega_{h_n}} \phi_{i,h_n} d\mu = \int_{\Omega_{h_n}} \sum_{i=1}^{k_{h_n}} (f(\mathbf{A}_i) \phi_{i,h_n}) d\mu,$$

where $\{\mathbf{A}_i\}_{i=1}^{k_{h_n}}$ are the vertexes of \mathcal{T}_{h_n} contained in the interior of Ω_{h_n} . By (5.2c) and the construction of $\phi_{i,h}$ in (5.5), it is easy to see that

$$\lim_{n \rightarrow +\infty} \sum_{i=1}^{k_{h_n}} (f(\mathbf{A}_i) \phi_{i,h_n})(\mathbf{x}) = f(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega, \quad \text{and} \quad \sup_{\mathbf{x} \in \Omega_{h_n}} \left| \sum_{i=1}^{k_{h_n}} (f(\mathbf{A}_i) \phi_{i,h_n})(\mathbf{x}) \right| \leq \sup_{\mathbf{x} \in \Omega} |f(\mathbf{x})|.$$

Then by dominated convergence Theorem, we know that

$$\int_{\Omega_{h_n}} f d\mu_h \rightarrow \int_{\Omega} f d\mu, \quad \text{as } n \rightarrow +\infty.$$

This implies that

$$\int_{\Omega} f d\mu_0 = \int_{\Omega} f d\mu, \quad \forall f \in C_c^0(\Omega).$$

Thus, we have that

$$\int_{\partial u_0(e)} R(\mathbf{p}) d\mathbf{p} = \mu_0(e) = \mu(e) \quad \forall \text{ Borel set } e \subset \Omega.$$

We denote by b_0 the border of u_0 , which is given by

$$b_0(\mathbf{x}) = \liminf_{\Omega \ni \mathbf{x}' \rightarrow \mathbf{x}} u_0(\mathbf{x}'), \quad \forall \mathbf{x} \in \partial\Omega.$$

By Lemma 3.2 and the fact that $g \in C(\partial\Omega)$, it is sufficient to show

$$b_0 = g \text{ on } \partial\Omega. \quad (6.1)$$

In fact, (6.1) would be an immediate consequence if we apply Theorem 4.10 to $\{u_{h_n}\}_{n=1}^{+\infty}$ and u_0 . In the following, we only need to verify all assumptions of Theorem 4.10 hold. Obviously, from our assumptions, Assumptions 4.1 and 4.2 hold. By (5.5) and the construction of u_h in (5.4), we know that for any $n \in \mathbb{N}$,

$$\int_{\partial u_{h_n}(e \cap \Omega_{h_n})} R(\mathbf{p}) d\mathbf{p} = \int_{e \cap \Omega_{h_n}} \left(\sum_{i=1}^{k_{h_n}} \phi_{i,h_n} \right) d\mu \leq \mu(e), \quad \forall \text{ Borel set } e \subset \Omega.$$

By Assumption 6.1, it is easy to check that Assumption 4.3 holds for $\{u_{h_n}\}_{n=1}^{+\infty}$ and u_0 . Moreover, by Lemma 6.2 and the construction of u_h in (5.4), Assumption 4.4 is also valid for $\{u_{h_n}\}_{n=1}^{+\infty}$ and u_0 . Thus all the assumptions of Theorem 4.10 hold. Then we have (6.1). \square

Theorem 6.5. *Let all the assumptions of Lemma 6.4 hold. Then, (1.1) admits a unique function $u \in W^+(\Omega) \cap C(\overline{\Omega})$. In addition, for any $\delta > 0$, there holds:*

$$\lim_{I \ni h \rightarrow 0} \|u_h - u\|_{L^\infty(\overline{\Omega_\delta})} = 0 \quad (6.2)$$

where $\Omega_\delta = \{\mathbf{x} \in \Omega : \text{dist}(\mathbf{x}, \partial\Omega) > \delta\}$ and for any $h \in I$, u_h is the solution of the finite element method (5.4).

Proof. By Lemma 6.4, we know that (1.1) admits a solution $u_0 \in W^+(\Omega) \cap C(\overline{\Omega})$. By Theorem 2.1, we know that u_0 is the unique solution to (1.1).

We shall prove (6.2) by contradiction. If (6.2) is not true, then there is $\delta_0 > 0$ and $\{h'_n\}_{n=1}^{+\infty} \subset I$ with $\lim_{n \rightarrow +\infty} h'_n = 0$, such that

$$\lim_{n \rightarrow +\infty} \|u_{h'_n} - u_0\|_{L^\infty(\overline{\Omega_{\delta_0}})} \neq 0.$$

By applying Lemma 6.4 to $\{u_{h'_n}\}_{n=1}^{+\infty}$, we know that there exist a function $u'_0 \in W^+(\Omega) \cap C(\overline{\Omega})$ satisfying (1.1), and a subsequence of $\{u_{h'_n}\}_{n=1}^{+\infty}$, still denoted by $\{u_{h'_n}\}_{n=1}^{+\infty}$, such that $u_{h'_n}$ converges to u'_0 uniformly on $\overline{\Omega_{\delta_0}}$ as $n \rightarrow +\infty$. Since (1.1) admits a unique solution, then $u_0 = u'_0$. This is a contradiction. Therefore, (6.2) is true. \square

7. GENERALIZED SOLUTION WITH DIRICHLET DATA IMPOSED WEAKLY

In this section, we firstly introduce the finite element method (7.1) for solving (1.2), which is based on the finite element method (5.4). Then we show that (1.2) is well-posed and the solutions of (7.1) converge to the exact solution. The main result in this section is Theorem 7.1.

The finite element method for (1.2) is to find $u_h^\delta \in H_h$ such that

$$\int_{\partial u_h^\delta(A_i)} R(\mathbf{p}) d\mathbf{p} = \int_{\Omega_h} \phi_{i,h} d\mu^\delta, \quad \forall 1 \leq i \leq k_h. \quad (7.1)$$

Here, for any $\delta > 0$, μ^δ is a measure defined by $\mu^\delta(e) := \mu(e \cap \Omega_\delta)$ for any Borel set $e \subset \Omega$, H_h is defined in (5.3) and $\phi_{i,h}$ is introduced in (5.5).

Theorem 7.1. *Let Assumptions 4.1, 4.2, 5.1 hold for the domain Ω and the function R . Then there is a unique function $u \in W^+(\Omega)$ satisfying (1.2) and (1.3). In addition, for any $\sigma > 0$,*

$$\lim_{\delta \rightarrow 0^+} \left(\lim_{h \rightarrow 0, h \in I} \|u_h^\delta - u\|_{L^\infty(\overline{\Omega_\sigma})} \right) = 0, \quad (7.2)$$

where $\Omega_\sigma = \{\mathbf{x} \in \Omega : \text{dist}(\mathbf{x}, \partial\Omega) > \sigma\}$.

Proof. For any $\delta > 0$, we look for $u^\delta \in W^+(\Omega) \cap C^0(\overline{\Omega})$ satisfying

$$\begin{cases} \int_{\partial u^\delta(e)} R(\mathbf{p}) d\mathbf{p} = \mu^\delta(e) & \forall \text{ Borel set } e \subset \Omega, \\ u^\delta = g & \text{on } \partial\Omega. \end{cases} \quad (7.3)$$

It is easy to check that Assumption 6.1 holds for μ^δ with λ large enough. Thus, one obtains that $k < K$ where

$$K = \frac{d + \tau + 1}{\tau + 2} + \frac{\lambda}{2}.$$

Then by Theorem 6.5, there is a unique function $u^\delta \in W^+(\Omega) \cap C^0(\overline{\Omega})$ satisfying (7.3). By [3, Theorem 10.4], it is easy to see that

$$\min_{\mathbf{y} \in \partial\Omega} g(\mathbf{y}) - \left(\sup_{\mathbf{x}, \mathbf{x}' \in \Omega} |\mathbf{x} - \mathbf{x}'| \right) \cdot g_R^{-1} \left(\int_{\Omega} R(\mathbf{p}) d\mathbf{p} \right) \leq u^\delta(\mathbf{x}) \leq \max_{\mathbf{y} \in \partial\Omega} g(\mathbf{y}), \quad \forall \mathbf{x} \in \Omega. \quad (7.4)$$

Here,

$$g_R(\rho) := \int_{B_\rho(\mathbf{0})} R(\mathbf{p}) d\mathbf{p}, \quad \forall \rho > 0.$$

By Theorem 2.1, for any $0 < \delta' < \delta$, it holds:

$$u^{\delta'}(\mathbf{x}) \leq u^\delta(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega. \quad (7.5)$$

By (7.4)-(7.5), we know that $\lim_{\delta \rightarrow 0^+} u^\delta(\mathbf{x})$ exists for any $\mathbf{x} \in \Omega$. Then we define

$$u(\mathbf{x}) := \lim_{\delta \rightarrow 0^+} u^\delta(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega.$$

Obviously, $u \in W^+(\Omega)$ and for any $\delta > 0$, it holds: $u(\mathbf{x}) \leq u^\delta(\mathbf{x})$, $\forall \mathbf{x} \in \Omega$. Since $u^\delta|_{\partial\Omega} = g$ for any $\delta > 0$, then we obtain

$$\limsup_{\Omega \ni \mathbf{x}' \rightarrow \mathbf{x}} u(\mathbf{x}') \leq g(\mathbf{x}) \quad \forall \mathbf{x} \in \partial\Omega.$$

By Theorem 4.1, (7.4) and (7.5), we get that

$$\begin{cases} \lim_{\delta \rightarrow 0^+} \|u^\delta - u\|_{L^\infty(\overline{\Omega_\sigma})} = 0, & \forall \sigma > 0, \\ \int_{\partial u(e)} R(\mathbf{p}) d\mathbf{p} = \mu(e) & \forall e \text{ a Borel set of } \Omega. \end{cases}$$

Thus u satisfies (1.2).

For any function $v \in W^+(\Omega)$ satisfies (1.2), we know, by Theorem 2.1, that $v(\mathbf{x}) \leq u^\delta(\mathbf{x})$ for all $\mathbf{x} \in \Omega$ and $\delta > 0$, which implies that $v(\mathbf{x}) \leq u(\mathbf{x})$, $\forall \mathbf{x} \in \Omega$. Therefore, u satisfies (1.3).

Finally, by Applying Theorem 6.5 to u^δ and u_h^δ , we obtain (7.2) immediately. \square

APPENDIX A. PROOF OF THEOREM 4.10

Proof. We follow the original proof of [3, Theorem 10.6], which consists of three parts. In the following, we give detailed explanation of the first part, which is very geometrically intuitive. Then, we give revision to the second part due to Assumption 4.2, which is weaker than [3, Assumption 10.1]. Finally, in the third part we explain why the revision made in the second part does not affect the analysis.

Part 1. Suppose that b_0 does not coincide with \tilde{b} on $\partial\Omega$. By Lemma 4.8, $b_0 \leq \tilde{b}$ on $\partial\Omega$. Then, there is $\mathbf{x}_0 \in \partial\Omega$ such that

$$b_0(\mathbf{x}_0) < \tilde{b}(\mathbf{x}_0).$$

Now we introduce special Cartesian coordinates in \mathbb{R}^d and \mathbb{R}^{d+1} : the axes x^1, \dots, x^{d-1} are in the supporting $(d-1)$ -dimensional plane α of $\partial\Omega$ at the point \mathbf{x}_0 , the axis x^d is orthogonal to α , and finally axis z is orthogonal to the hyperplane \mathbb{R}^d . For simplicity, we take the point $\mathbf{x}_0 \in \mathbb{R}^d$ to be $\mathbf{0} \in \mathbb{R}^d$ and we define two points in \mathbb{R}^{d+1} by

$$\mathbf{Q} := (0, \dots, 0, \tilde{b}(\mathbf{0})) \text{ and } \overline{\mathbf{Q}} := (0, \dots, 0, b_0(\mathbf{0}))$$

and for any $0 < \delta < 1$, we introduce two new points in \mathbb{R}^{d+1}

$$\mathbf{Q}' := (0, \dots, 0, \tilde{b}(\mathbf{0}) - \delta\Delta l) \text{ and } \mathbf{Q}'' := (0, \dots, 0, b_0(\mathbf{0}) - \delta\Delta l),$$

where $\Delta l := \tilde{b}(\mathbf{0}) - b_0(\mathbf{0})$. Obviously, \mathbf{Q} , $\overline{\mathbf{Q}}$, \mathbf{Q}' and \mathbf{Q}'' are all along the z -axis. Now consider two hyperplanes β' and β'' in \mathbb{R}^{d+1} by equations:

$$\begin{cases} \beta' & : & z = \tilde{b}(\mathbf{0}) - \delta\Delta l - \gamma^{-1}x^d, \\ \beta'' & : & z = b_0(\mathbf{0}) - \delta\Delta l, \end{cases}$$

where γ is a sufficient small positive number.

The task in part 1 is to show (A.11) holds. That is, we need to prove that

$$\int_{\partial V(\overline{\mathbf{Q}})} R(\mathbf{p}) d\mathbf{p} \leq d_2 \gamma^{\lambda + \frac{d+\tau+1}{\tau+2}}.$$

Here V is the convex cone with the vertex o $\overline{\mathbf{Q}}$ and the basis $\beta' \cap \mathbf{K}$ and \mathbf{K} is defined in (A.1).

We would like to point out that with respect to these three numbers δ, γ and n , $\delta \rightarrow 0$ implies that $\gamma \rightarrow 0$, and $\gamma \rightarrow 0$ implies that $n \rightarrow +\infty$ in the following analysis. In part 1 of the proof, we choose $0 < \delta < 1$ arbitrarily, $\gamma > 0$ small enough, and n large enough.

Let $Z = \partial\Omega \times \mathbb{R} \subset \mathbb{R}^{d+1}$. Then Z bounds some convex body \mathbf{K} together with the hyperplanes β' and β'' . Here the convex body \mathbf{K} is define by

$$\mathbf{K} := \{(\mathbf{x}, z) \in \overline{\Omega} \times \mathbb{R} : b_0(\mathbf{0}) - \delta\Delta l \leq z \leq \tilde{b}(\mathbf{0}) - \delta\Delta l - \gamma^{-1}x^d\}. \quad (\text{A.1})$$

It is easy to see that \mathbf{K} is a closed set in \mathbb{R}^{d+1} and $\text{Int}(\mathbf{K}) \neq \emptyset$ for any $\delta, \gamma > 0$. We define

$$H(\mathbf{K}) := \{\mathbf{x} \in \mathbb{R}^d : \exists z \in \mathbb{R} \text{ such that } (\mathbf{x}, z) \in \mathbf{K}\},$$

which is the projection of \mathbf{K} on \mathbb{R}^d . If $\gamma > 0$ is small enough, we know that

$$H(\mathbf{K}) = \{\mathbf{x} \in \overline{\Omega} : x^d \leq \bar{x}^d\}, \text{ with } \bar{x}^d := \gamma(\tilde{b}(\mathbf{0}) - b_0(\mathbf{0})).$$

According to Lemma 3.2 and part (a) of Assumption 4.4, for any $n \in \mathbb{N}$, v_n can be extended continuously to $\partial\Omega_n$ such that $v_n(\mathbf{x}) = b_n(\mathbf{x})$, $\forall \mathbf{x} \in \partial\Omega_n$. We define

$$S_{v_n} := \{(\mathbf{x}, v_n(\mathbf{x})) : \mathbf{x} \in \overline{\Omega_n}\}, \quad \forall n \in \mathbb{N}.$$

In the following, we give five important claims (A.2) - (A.6):

1) for any given $\gamma > 0$, it holds:

$$\text{Int}(S_{v_n}) \cap \text{Int}(\mathbf{K}) \neq \emptyset, \text{ with } n \text{ large enough.} \quad (\text{A.2})$$

In fact, due to the definition of b_0 , there is $\mathbf{x} \in \text{Int}(H(\mathbf{K}))$ satisfying

$$b_0(\mathbf{0}) - \delta\Delta l < v_0(\mathbf{x}) < b_0(\mathbf{0}) + \frac{1}{2}(1 - \delta)\Delta l.$$

Since $b_0(\mathbf{0}) + \frac{1}{2}(1 - \delta)\Delta l < \tilde{b}(\mathbf{0}) - \delta\Delta l$, we can choose $\mathbf{x} \in \text{Int}(H(\mathbf{K}))$ close enough to $\mathbf{0}$ such that

$$b_0(\mathbf{0}) - \delta\Delta l < v_0(\mathbf{x}) < \tilde{b}(\mathbf{0}) - \delta\Delta l - \gamma^{-1}x^d.$$

Since $v_n(\mathbf{x})$ converges to $v_0(\mathbf{x})$ as $n \rightarrow +\infty$, then if n is large enough, it holds:

$$b_0(\mathbf{0}) - \delta\Delta l < v_n(\mathbf{x}) < \tilde{b}(\mathbf{0}) - \delta\Delta l - \gamma^{-1}x^d.$$

Then, $(\mathbf{x}, v_n(\mathbf{x})) \in \text{Int}(S_{v_n}) \cap \text{Int}(\mathbf{K})$, if n is large enough. Therefore the claim (A.2) holds true.

2) If $\gamma > 0$ is small enough and n is large enough, it holds:

$$\partial\mathbf{K} \cap S_{v_n} \subset (\partial\mathbf{K} \cap \beta') \setminus Z. \quad (\text{A.3})$$

In fact, we can easily see that $\partial \mathbf{K} = \Gamma \cup \tilde{H}(\mathbf{K}) \cup (\partial \mathbf{K} \cap \beta')$, where

$$\left\{ \begin{array}{l} \Gamma := \{(\mathbf{x}, z) \in \mathbb{R}^{d+1} : \mathbf{x} \in \partial H(\mathbf{K}), b_0(\mathbf{0}) - \delta \Delta l < z < \tilde{b}(\mathbf{0}) - \delta \Delta l - \gamma^{-1} x^d\}, \\ \tilde{H}(\mathbf{K}) := \{(\mathbf{x}, b_0(\mathbf{0}) - \delta \Delta l) : \mathbf{x} \in H(\mathbf{K})\}. \end{array} \right.$$

If $\gamma > 0$ is small enough, then $\Omega \cap \{\mathbf{x} \in \mathbb{R}^d : x^d = \bar{x}^d\} \neq \emptyset$. Then we have

$$\left\{ \begin{array}{l} H(\mathbf{K}) = \{\mathbf{x} \in \bar{\Omega} : x^d \leq \bar{x}^d\}, \quad \tilde{H}(\mathbf{K}) = \{(\mathbf{x}, b_0(\mathbf{0}) - \delta \Delta l) : \mathbf{x} \in \bar{\Omega}, \quad x^d \leq \bar{x}^d\}, \\ \Gamma = \{(\mathbf{x}, z) \in \mathbb{R}^{d+1} : \mathbf{x} \in \partial \Omega, \quad x^d < \bar{x}^d, \quad b_0(\mathbf{0}) - \delta \Delta l < z < \tilde{b}(\mathbf{0}) - \delta \Delta l - \gamma^{-1} x^d\}, \\ \bar{\Gamma} = \{(\mathbf{x}, z) \in \mathbb{R}^{d+1} : \mathbf{x} \in \partial \Omega, \quad x^d \leq \bar{x}^d, \quad b_0(\mathbf{0}) - \delta \Delta l \leq z \leq \tilde{b}(\mathbf{0}) - \delta \Delta l - \gamma^{-1} x^d\}. \end{array} \right.$$

The claim (A.2) follows directly if we can prove that $S_{v_n} \cap \bar{\Gamma} = S_{v_n} \cap \tilde{H}(\mathbf{K}) = \emptyset$ if $\gamma > 0$ is small enough and n is large enough.

Firstly, we show that $S_{v_n} \cap \bar{\Gamma} = \emptyset$ if $\gamma > 0$ is small enough and n is large enough. If not, then there is a subsequence of \mathbb{N} , still denote by \mathbb{N} , for simplicity, such that $S_{v_n} \cap \bar{\Gamma} \neq \emptyset$ for any $n \in \mathbb{N}$. Then for any $n \in \mathbb{N}$, there is $\mathbf{x}_n \in \partial \Omega_n \cap \Omega$, such that $x_n^d \leq \bar{x}^d$, and $b_0(\mathbf{0}) - \delta \Delta l \leq v_n(\mathbf{x}_n) \leq \tilde{b}(\mathbf{0}) - \delta \Delta l - \gamma^{-1} x_n^d$. According to Assumption 4.1, $\lim_{\gamma \rightarrow 0}^T H(\mathbf{K}) = \mathbf{0} \in \mathbb{R}^d$. From (b) of Assumption 4.4, we obtain

$$\sup_{\mathbf{x} \in \partial \Omega, \quad x^d \leq \bar{x}^d} |\tilde{b}(\mathbf{x}) - \tilde{b}(\mathbf{0})| \rightarrow 0, \quad \text{as } \gamma \rightarrow 0.$$

Hence we can choose $\gamma > 0$ small enough, such that it holds:

$$|\tilde{b}(\mathbf{x}) - \tilde{b}(\mathbf{0})| < \frac{1}{3} \delta \Delta l, \quad \forall \mathbf{x} \in \{\mathbf{y} \in \partial \Omega : y^d \leq \bar{x}^d\}.$$

Since $\mathbf{x}_n \in \{\mathbf{y} \in \partial \Omega : y^d \leq \bar{x}^d\}$, then we get

$$|\tilde{b}(\mathbf{x}_n) - \tilde{b}(\mathbf{0})| < \frac{1}{3} \delta \Delta l, \quad \forall n \in \mathbb{N}.$$

According to part (b) of Assumption 4.4, there is a subsequence of \mathbb{N} which we still denote by \mathbb{N} for the sake of simplicity, such that

$$(\mathbf{x}_n, v_n(\mathbf{x}_n)) = (\mathbf{x}_n, b_n(\mathbf{x}_n)) \rightarrow (\tilde{\mathbf{x}}, \tilde{b}(\tilde{\mathbf{x}})), \quad \text{as } n \rightarrow +\infty$$

for some point $\tilde{\mathbf{x}} \in \{\mathbf{y} \in \partial \Omega : y^d \leq \bar{x}^d\}$. Then if $n \in \mathbb{N}$ large enough,

$$|v_n(\mathbf{x}_n) - \tilde{b}(\mathbf{0})| < \frac{2}{3} \delta \Delta l,$$

which implies that if $n \in \mathbb{N}$ large enough,

$$v_n(\mathbf{x}_n) > \tilde{b}(\mathbf{0}) - \frac{2}{3} \delta \Delta l > \tilde{b}(\mathbf{0}) - \delta \Delta l - \gamma^{-1} x_n^d.$$

This is a contradiction. Thus $S_{v_n} \cap \bar{\Gamma} = \emptyset$ for $\gamma > 0$ small enough and n large enough.

Secondly, we show that $S_{v_n} \cap \tilde{H}(\mathbf{K}) = \emptyset$ if $\gamma > 0$ is small enough and n is large enough. According to Assumption 4.1, we know that $\lim_{\gamma \rightarrow 0}^T H(\mathbf{K}) = \mathbf{0} \in \mathbb{R}^d$. By the definition of b_0 , we can see that if $\gamma > 0$ is small enough,

$$v_0(\mathbf{x}) > b_0(\mathbf{0}) - \frac{1}{6} \delta \Delta l, \quad \forall \mathbf{x} \in \text{Int}(H(\mathbf{K}))$$

and there is a point $\tilde{\mathbf{x}} \in H(\mathbf{K}) \cap \Omega$ such that

$$|v_0(\tilde{\mathbf{x}}) - b_0(\mathbf{0})| < \frac{1}{6}\delta\Delta l.$$

Since $\bar{x}^d := \gamma(\tilde{b}(\mathbf{0}) - b_0(\mathbf{0}))$ and $H(\mathbf{K}) = \{\mathbf{x} \in \bar{\Omega} : x^d \leq \bar{x}^d\}$ for $\gamma > 0$ small enough, we can choose $\tilde{\mathbf{x}}$ satisfying $\tilde{x}^d = \bar{x}^d$ if $\gamma > 0$ is small enough. Obviously, $\tilde{\mathbf{x}} \in \text{Int}(\{\mathbf{x} \in \partial H(\mathbf{K}) : x^d = \bar{x}^d\})$. We define

$E(\mathbf{K}) := \{\mathbf{x} \in H(\mathbf{K}) : \exists \mathbf{y} \in \partial\Omega \text{ with } y^d \leq \bar{x}^d \text{ such that } \mathbf{x} \text{ is in the line segment}$

$$\text{between } \mathbf{y} \text{ and } \tilde{\mathbf{x}}, \text{ and } \text{dist}(\mathbf{x}, \tilde{\mathbf{x}}) \leq \frac{1}{2}\text{dist}(\mathbf{y}, \tilde{\mathbf{x}})\}.$$

It is easy to see that $E(\mathbf{K})$ is a closed subset of Ω and $\text{dist}(\partial\Omega, E(\mathbf{K})) > 0$. From [3, Lemma 3.1], (4.2) and part (a) of Assumption 4.3, we get

$$\lim_{n \rightarrow +\infty} \|v_n - v_0\|_{L^\infty(E(\mathbf{K}))} = 0,$$

which implies that if n is large enough,

$$v_n(\tilde{\mathbf{x}}) < b_0(\mathbf{0}) + \frac{1}{4}\delta\Delta l, \text{ and } v_n(\mathbf{x}) > b_0(\mathbf{0}) - \frac{1}{4}\delta\Delta l, \quad \forall \mathbf{x} \in E(\mathbf{K}).$$

Due to the fact that $v_n \in W^+(\Omega_n)$ and the definition of $E(\mathbf{K})$, we know that if n is large enough,

$$v_n(\mathbf{x}) > b_0(\mathbf{0}) - \frac{3}{4}\delta\Delta l, \quad \forall \mathbf{x} \in H(\mathbf{K}) \cap \Omega_n.$$

Thus if n is large enough, $v_n(\mathbf{x}) > b_0(\mathbf{0}) - \delta\Delta l$, $\forall \mathbf{x} \in H(\mathbf{K}) \cap \bar{\Omega}_n$, which shows that $S_{v_n} \cap \tilde{H}(\mathbf{K}) = \emptyset$ if $\gamma > 0$ is small enough and n is large enough.

3) If $\gamma > 0$ is small enough and $n \in \mathbb{N}$ is large enough,

$$\exists \mathbf{Q}_n \in \text{Int} \mathbf{K} \cap \text{Int}(S_{v_n}) \text{ such that } \lim_{n \rightarrow +\infty} \text{dist}(\bar{\mathbf{Q}}, \mathbf{Q}_n) = 0. \quad (\text{A.4})$$

In fact, $\forall \epsilon > 0$, by the definitions of b_0 and $H(\mathbf{K})$, there is $\mathbf{x} \in \text{Int}(H(\mathbf{K}))$ such that $|\mathbf{x}| < \epsilon$ and there holds:

$$-\frac{1}{2}\min(\epsilon, \delta\Delta l) < v_0(\mathbf{x}) - b_0(\mathbf{0}) < \frac{1}{2}\min(\epsilon, (1 - \delta)\Delta l - \gamma^{-1}x^d).$$

By part (a) of Assumption 4.3, for any $n \in \mathbb{N}$ large enough,

$$-\min(\epsilon, \delta\Delta l) < v_n(\mathbf{x}) - b_0(\mathbf{0}) < \min(\epsilon, (1 - \delta)\Delta l - \gamma^{-1}x^d).$$

Thus for any $n \in \mathbb{N}$ large enough, we can infer

$$\text{dist}((\mathbf{x}, v_n(\mathbf{x})), \bar{\mathbf{Q}}) < \epsilon \text{ and } (\mathbf{x}, v_n(\mathbf{x})) \in \text{Int}(K).$$

By (4.2), $\mathbf{x} \in \Omega_n$ if $n \in \mathbb{N}$ is large enough. Therefor, (A.4) holds true.

4) If $\gamma > 0$ is small enough and $n \in \mathbb{N}$ is large enough,

$$\partial V_n(\mathbf{Q}_n) \subset \partial v_n(\text{Int}(H_n(\mathbf{K}))), \quad (\text{A.5})$$

where \mathbf{Q}_n is defined in (A.4) and

$$\begin{cases} S_n(\mathbf{K}) := S_{v_n} \cap \mathbf{K}, & \beta'(\mathbf{K}) := \beta' \cap \mathbf{K}, \\ H_n(\mathbf{K}) := \{\mathbf{x} \in H(\mathbf{K}) : \exists z \in \mathbb{R} \text{ such that } (\mathbf{x}, z) \in S_n(\mathbf{K})\}, \\ V_n \text{ is the convex cone with the vertex } \mathbf{Q}_n \text{ and the base } \beta'(\mathbf{K}). \end{cases}$$

In fact, by (A.3), we see that

$$H_n(\mathbf{K}) = \{\mathbf{x} \in H(\mathbf{K}) \cap \overline{\Omega_n} : \tilde{b}(\mathbf{0}) - \delta\Delta l - \gamma^{-1}x^d \geq v_n(\mathbf{x})\}.$$

for $\gamma > 0$ small enough and $n \in \mathbb{N}$ large enough.

We define functions \tilde{V}_n in $H(\mathbf{K})$ by

$$\tilde{V}_n(\mathbf{x}) = \inf_{z \in \mathbb{R}} (\mathbf{x}, z) \in V_n, \quad \forall \mathbf{x} \in H(\mathbf{K}).$$

Then $\tilde{V}_n \in W^+(H(\mathbf{K}))$. Furthermore, by (A.3, A.4), we can see that $\mathbf{Q}_n \in S_n(\mathbf{K}) \cap V_n$, and $\tilde{V}_n(\mathbf{x}) \leq v_n(\mathbf{x})$, $\forall \mathbf{x} \in \partial H_n(\mathbf{K})$ if $\gamma > 0$ is small enough and $n \in \mathbb{N}$ is large enough.

Let T be a supporting hyperplane of V_n at \mathbf{Q}_n with the equation

$$z = z_T + \mathbf{p}_T \cdot \mathbf{x}.$$

Let $(\mathbf{x}_n, z_n) = \mathbf{Q}_n$ where $\mathbf{x}_n \in \mathbb{R}^d$. Then we obtain

$$\begin{cases} v_n(\mathbf{x}_n) = z_T + \mathbf{p}_T \cdot \mathbf{x}_n, \\ v_n(\mathbf{x}) \geq z_T + \mathbf{p}_T \cdot \mathbf{x}, \quad \forall \mathbf{x} \in \partial H_n(\mathbf{K}). \end{cases}$$

By (A.4), we know that $\tilde{b}(\mathbf{0}) - \delta\Delta l - \gamma^{-1}x_n^d > z_n = v_n(\mathbf{x}_n)$. Thus $\mathbf{x}_n \in \text{Int}(H_n(\mathbf{K}))$. In the following, we shall show that $\mathbf{p}_T \in \partial v_n(\text{Int}(H_n(\mathbf{K})))$. In fact, if $v_n(\mathbf{x}) \geq z_T + \mathbf{p}_T \cdot \mathbf{x}$, $\forall \mathbf{x} \in H_n(\mathbf{K})$, then $\mathbf{p}_T \in \partial v_n(\text{Int}(H_n(\mathbf{K})))$. On the other hand, if $\{\mathbf{x} \in H_n(\mathbf{K}) : v_n(\mathbf{x}) < z_T + \mathbf{p}_T \cdot \mathbf{x}\} \neq \emptyset$, then by the fact

$$v_n(\mathbf{x}) \geq z_T + \mathbf{p}_T \cdot \mathbf{x}, \quad \forall \mathbf{x} \in \partial H_n(\mathbf{K}),$$

we know that

$$\{\mathbf{x} \in H_n(\mathbf{K}) : \tilde{V}_n(\mathbf{x}) < z_T + \mathbf{p}_T \cdot \mathbf{x}\} \subset \text{Int} H_n(\mathbf{K}).$$

By [11, Lemma 1.4.1], $\mathbf{p}_T \in \partial v_n(\text{Int}(H_n(\mathbf{K})))$. Thus (A.5) holds true.

5) If $\gamma > 0$ is small enough and $n \in \mathbb{N}$ is large enough

$$\int_{\partial V(\overline{\mathbf{Q}})} R(\mathbf{p}) d\mathbf{p} \leq \liminf_{n \rightarrow +\infty} \int_{\partial V_n(\mathbf{Q}_n)} R(\mathbf{p}) d\mathbf{p}, \quad (\text{A.6})$$

where V is a convex cone, with the vertex $\overline{\mathbf{Q}}$ and the basis $\beta'(\mathbf{K})$.

In fact, since $\text{Int}(H_n(\mathbf{K})) \subset H(\mathbf{K}) \cap \Omega_n$, then

$$\int_{\partial v_n(\text{Int}(H_n(\mathbf{K})))} R(\mathbf{p}) d\mathbf{p} \leq \int_{\partial v_n(H(\mathbf{K}) \cap \Omega_n)} R(\mathbf{p}) d\mathbf{p},$$

which, by claim (A.5), shows that

$$\int_{\partial V_n(\mathbf{Q}_n)} R(\mathbf{p}) d\mathbf{p} \leq \int_{\partial v_n(H(\mathbf{K}) \cap \Omega_n)} R(\mathbf{p}) d\mathbf{p}. \quad (\text{A.7})$$

We know that

$$\begin{cases} \int_{\partial V_n(\mathbf{Q}_n)} R(\mathbf{p}) d\mathbf{p} = \int_{\mathbb{R}^d} \chi_{\partial V_n(\mathbf{Q}_n)}(\mathbf{p}) R(\mathbf{p}) d\mathbf{p}, \\ \int_{\partial V(\overline{\mathbf{Q}})} R(\mathbf{p}) d\mathbf{p} = \int_{\mathbb{R}^d} \chi_{\partial V(\overline{\mathbf{Q}})}(\mathbf{p}) R(\mathbf{p}) d\mathbf{p}, \end{cases}$$

where $\chi_{\partial V_n(\mathbf{Q}_n)}$ and $\chi_{\partial V(\overline{\mathbf{Q}})}$ are characteristic functions. According to Fatou's Lemma,

$$\int_{\mathbb{R}^d} \liminf_{n \rightarrow +\infty} (\chi_{\partial V_n(\mathbf{Q}_n)}(\mathbf{p}) R(\mathbf{p})) d\mathbf{p} \leq \liminf_{n \rightarrow +\infty} \int_{\mathbb{R}^d} \chi_{\partial V_n(\mathbf{Q}_n)}(\mathbf{p}) R(\mathbf{p}) d\mathbf{p}.$$

Hence, to prove (A.7), it is sufficient to show that

$$\liminf_{n \rightarrow +\infty} \chi_{\partial V_n(\mathbf{Q}_n)}(\mathbf{p}) \geq \chi_{\partial V(\overline{\mathbf{Q}})}(\mathbf{p}), \quad \forall \text{ a.e. } \mathbf{p} \in \mathbb{R}^d. \quad (\text{A.8})$$

Let T be a supporting hyperplane of V at $\overline{\mathbf{Q}}$ with

$$z = z_T + \mathbf{p}_T \cdot \mathbf{x}.$$

We notice that the projection of $\beta'(\mathbf{K})$ onto \mathbb{R}^d is $H(\mathbf{K})$ and $\overline{\mathbf{Q}}$ is the vertex of the convex cone V . Then we know that

$$\mathbf{p}_T \in \text{Int}(\partial V(\overline{\mathbf{Q}})) \iff \tilde{b}(\mathbf{0}) - \delta \Delta l - \gamma^{-1} x^d > z_T + \mathbf{p}_T \cdot \mathbf{x}, \quad \forall \mathbf{x} \in \partial H(\mathbf{K}).$$

We define

$$\epsilon_0 := \inf_{\mathbf{x} \in \partial H(\mathbf{K})} ((\tilde{b}(\mathbf{0}) - \delta \Delta l - \gamma^{-1} x^d) - (z_T + \mathbf{p}_T \cdot \mathbf{x})).$$

Then $\epsilon_0 > 0$. For any $n \in \mathbb{N}$, we choose $z_n \in \mathbb{R}$ such that

$$z = z_n + \mathbf{p}_T \cdot \mathbf{x}$$

is a hyperplane passing through \mathbf{Q}_n . By (A.4) and the fact that $\epsilon_0 > 0$, it holds:

$$\tilde{b}(\mathbf{0}) - \delta \Delta l - \gamma^{-1} x^d \geq z_n + \mathbf{p}_T \cdot \mathbf{x}, \quad \forall \mathbf{x} \in \partial H(\mathbf{K}).$$

if $\gamma > 0$ is small enough and $n \in \mathbb{N}$ is large enough. Then $\mathbf{p}_T \in \partial V_n(\mathbf{Q}_n)$, which shows that (A.8) holds. Thus (A.6) is true.

In the following, we finish the proof of part 1. According to Assumption 4.1, if $\gamma > 0$ is small enough, the Borel set $H(\mathbf{K}) \subset U_\rho(\mathbf{0}) \cap \Omega$. By part (b) of Assumption 4.3,

$$\liminf_{n \rightarrow +\infty} \int_{\partial v_n(H(\mathbf{K}) \cap \Omega_n)} R(\mathbf{p}) d\mathbf{p} \leq C_1 \left(\sup_{\mathbf{x} \in H(\mathbf{K})} \text{dist}(\mathbf{x}, \partial \Omega) \right)^\lambda |H(\mathbf{K})|.$$

The last inequality with (A.6) implies the inequality

$$\int_{\partial V(\overline{\mathbf{Q}})} R(\mathbf{p}) d\mathbf{p} \leq C_1 \left(\sup_{\mathbf{x} \in H(\mathbf{K})} \text{dist}(\mathbf{x}, \partial \Omega) \right)^\lambda |H(\mathbf{K})|. \quad (\text{A.9})$$

Clearly we have

$$\sup_{\mathbf{x} \in H(\mathbf{K})} \text{dist}(\mathbf{x}, \partial\Omega) = \gamma\Delta l.$$

Let $P := \{\mathbf{x} \in \mathbb{R}^d : \eta(\mathbf{0})(\sum_{i=1}^{d-1}|x^i|^2)^{\frac{\tau+2}{2}} \leq x^d \leq \gamma\Delta l\}$, where $\eta(\mathbf{0})$ is the positive constant introduced in Definition 4.4. Hence $H(\mathbf{K}) \subset P$ and it holds:

$$\begin{aligned} |P| &= m_{d-1} \int_0^{\gamma\Delta l} \left(\frac{l}{\eta(\mathbf{0})}\right)^{\frac{d-1}{\tau+2}} dl \\ &= \frac{\tau+2}{d+1} m_{d-1} (\eta(\mathbf{0}))^{-\frac{d-1}{\tau+2}} (\gamma\Delta l)^{\frac{d+\tau+1}{\tau+2}} = d_1 \gamma^{\frac{d+\tau+1}{\tau+2}}. \end{aligned} \quad (\text{A.10})$$

From (A.9)- (A.10), we know that if $\gamma > 0$ is small enough,

$$\int_{\partial V(\bar{Q})} R(\mathbf{p}) d\mathbf{p} \leq d_2 \gamma^{\lambda + \frac{d+\tau+1}{\tau+2}}, \quad (\text{A.11})$$

where $d_2 = C_1 d_1 (\Delta l)^\lambda$ and d_1 is a positive constant depending only on given constants $0 \leq \tau < +\infty$, $\eta(\mathbf{0}) > 0$, Δl and m_{d-1} . (Notice that τ and $\eta(\mathbf{0})$ are introduced in (4.4), m_{d-1} is the volume of the unit $(d-1)$ -ball)

Part 2. According to [3, (10.45)] and [3, Lemma 10.4], We know that

$$\int_{\partial V(\bar{Q})} R(\mathbf{p}) d\mathbf{p} \geq \int_{H^d} R(\mathbf{p}) d\mathbf{p}. \quad (\text{A.12})$$

Here, H^d is the d -dimensional cone of revolution with axis p^d (axis in \mathbb{R}^d), vertex $(0, \dots, 0, -\frac{\delta}{\gamma}) \in \mathbb{R}^d$ and base H^{d-1} , which is the $(d-1)$ -dimensional ball given by the following equations: $|p^1|^2 + \dots + |p^{d-1}|^2 \leq (C')^2 \gamma^{-\frac{2}{\tau+2}}$, $p^d = -C'' \gamma^{-1}$ and the constants C' and C'' are given by

$$C' = \frac{\tau+2}{\tau+1} (1-\delta) (\Delta l)^{\frac{\tau+2}{\tau+1}} (\eta(\mathbf{0}))^{\frac{1}{\tau+2}}, \quad C'' = \frac{\tau+2-\delta}{\tau+1}.$$

Obviously, C' and C'' do not depend on γ and have positive limits as $\delta \rightarrow 0$.

The convex cone H^d lies between two parallel hyperplanes in \mathbb{R}^d

$$p^d = -\delta \gamma^{-1}, \quad p^d = -\frac{\tau+2-\delta}{(\tau+1)\gamma}.$$

In the original proof of [3, Theorem 10.6],

$$\int_{H^d} R(\mathbf{p}) d\mathbf{p} \geq C_0 \int_{H^d} |\mathbf{p}|^{-2k} d\mathbf{p}, \quad (\text{A.13})$$

due to [3, Assumption 10.1]. However, [3, Assumption 10.1] is so restrictive that the Gaussian curvature equation does not satisfy. Hence we use Assumption 4.2 instead.

The revision we made is to require two positive parameters δ and γ satisfying

$$\gamma \leq r_0^{-1} \delta \quad (\text{A.14})$$

where r_0 is the positive constant defined in Assumption 4.2.

If (A.14) is satisfied, then by (A.12, A.13), we know that

$$\int_{\partial V(\overline{Q})} R(\mathbf{p}) d\mathbf{p} \geq \int_{H^d} R(\mathbf{p}) d\mathbf{p} \geq C_0 \int_{H^d} |\mathbf{p}|^{-2k} d\mathbf{p}. \quad (\text{A.15})$$

Thus the remaining part of part 2 is the same as the second part of the proof of [3, Theorem 10.6] (right below the proof of [3, Lemma 10.4]).

Part 3. We completely follow the third part of the proof of [3, Theorem 10.6]. The task in part 3 is to show inequalities based on (A.15) does not hold as $\gamma > 0$ approaches to zero. We only need to verify inequalities [3, (10.68), (10.69), (10.74), (10.77), (10.78)] hold if (A.14) is satisfied. Thus the third part of the proof of [3, Theorem 10.6] can go through completely if (A.14) is satisfied. Therefore the proof is complete. \square

REFERENCES

- [1] A. D. Alexandrov, *Dirichlet's problem for the equation $\det\|z_{ij}\| = \phi(z_1, \dots, z_n, z, x_1, \dots, x_n)$* I. (Russian) Vestnik Leningrad. Univ. Ser. Mat. Meh. Astr. 13 (1958), 5-24.
- [2] I. Bakelman, *Generalized solutions of Monge-Ampère equations (Russian)*, Dokl. Akad. Nauk SSSR (N.S.) 114 (1957), 1143-1145.
- [3] I. Bakelman, *Convex analysis and nonlinear geometric elliptic equations*, Springer-Verlag, Berlin, 1994.
- [4] L. A. Caffarelli, *A localization property of viscosity solutions to the Monge-Ampère equation and their strict convexity*, Ann. of Math. (2) 131 (1990), 129-134.
- [5] L. A. Caffarelli, *$W^{2,p}$ estimates for solutions of the Monge-Ampère equation*, Ann. of Math. (2) 131 (1990), 135-150.
- [6] L. Caffarelli, L. Nirenberg, and J. Spruck, *The Dirichlet problem for nonlinear second-order elliptic equations. I. Monge-Ampère equation*, Comm. Pure Appl. Math. 37 (1984), no. 3, 369-402.
- [7] S. Y. Cheng and S. T. Yau, *On the regularity of the solution to the n -dimensional Minkowski problem*, Comm. Pure Appl. Math. 19 (1976), 595-516.
- [8] S. Y. Cheng and S. T. Yau, *On the regularity of the Monge-Ampère equation $\det(\partial^2 u / \partial x_i \partial x_j) = F(x, u)$* , Comm. Pure Appl. Math. 30 (1977), 41-68.
- [9] G. De Philippis and A. Figalli, *$W^{2,1}$ regularity for solutions of the Monge-Ampère equation*, Invent. Math. 192 (2013), 55-69.
- [10] G. De Philippis, A. Figalli, and O. Savin, *A note on interior $W^{2,1+\epsilon}$ estimates for the Monge-Ampère equation*, Math. Ann. 357 (2013), 11-22.
- [11] G.E. Gutiérrez, *The Monge-Ampère equation*, Birkhäuser, Boston, 2001.
- [12] N. M. Ivochkina, *Classical solvability of the Dirichlet problem for the Monge-Ampère equation*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 96 (1980), 69-79 (Russian); English translation in J. Soviet Math. 21 (1983), 689-697.
- [13] N. V. Krylov, *Boundedly nonhomogeneous elliptic and parabolic equations*, Izvestia, Math. Ser. 46, 1982, 487-523; English translation: Math. USSR Izvestia 20. 1983, pp. 459-492.
- [14] N. V. Krylov, *On degenerate nonlinear elliptic equations*. Mat. Shornik 120 (162). 1983, 311-330.
- [15] N. V. Krylov, *Boundedly nonhomogeneous elliptic and parabolic equations in a domain*, Izvestia Math. Ser. 47, 1983, 75-108.
- [16] F. Maggi, *Sets of finite perimeter and geometric variational problems*, Cambridge University Press, 2012.
- [17] O. Savin, *Pointwise $C^{2,\alpha}$ estimates at the boundary for the Monge-Ampère equation*, J. Amer. Math. Soc. 26 (2013), no. 1, 63-99

- [18] N. S. Trudinger and X. J. Wang, *Boundary regularity for the Monge-Ampère and affine maximal surface equations*, Ann. of Math. 167 (2008), 993-1028.
- [19] X. J. Wang, *Regularity for Monge-Ampère equation near the boundary*, Analysis 16 (1996), 101-107.
- [20] W.P. Ziemer, *Weakly differentiable functions*, Springer-Verlag, Berlin, 1989.

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